Appendix 1 to IV.10: Sommerfeld's expansion
Statistical Physics
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Consider integral,

$$I(\mu) = \int_0^\infty d\varepsilon \, \frac{\varphi(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}, \tag{1}$$

where $\varphi(\varepsilon)$ is some well behaved function. Let us introduce a new variable x,

$$x = \beta(\varepsilon - \mu), \quad \varepsilon = \mu + k_B T x. \tag{2}$$

The integral transforms to

$$I(\mu) = k_B T \int_{-\mu/k_B T}^\infty dx \, \frac{\varphi(\mu + k_B T x)}{e^x + 1} = k_B T \int_{-\mu/k_B T}^0 dx \, \frac{\varphi(\mu + k_B T x)}{e^x + 1} + k_B T \int_0^\infty dx \, \frac{\varphi(\mu + k_B T x)}{e^x + 1}. \tag{3}$$

Let us work little bit on the first integral on the right,

$$\int_{-\mu/k_B T}^0 dx \, \frac{\varphi(\mu + k_B T x)}{e^x + 1} = |x \to -x| = \int_0^{\mu/k_B T} dx \, \frac{\varphi(\mu - k_B T x)}{e^{-x} + 1} = \tag{4}$$

$$= \left| \frac{1}{e^{-x}+1} = 1 - \frac{1}{e^x + 1} \right| = \frac{1}{k_B T} \int_0^\mu dz \, \varphi(z) - \int_0^{\mu/k_B T} dx \, \frac{\varphi(\mu - k_B T x)}{e^x + 1}. \tag{5}$$

We have introduced new variable z, by

$$z = \mu - k_B T x, \tag{6}$$

in the first integral on the right. Our original integral now reads,

$$I(\mu) = \int_0^\mu dz \, \varphi(z) + k_B T \int_0^\infty dx \, \frac{\varphi(\mu + k_B T x)}{e^x + 1} - k_B T \int_0^{\mu/k_B T} dx \, \frac{\varphi(\mu - k_B T x)}{e^x + 1} \tag{7}$$

Thus far we have made no approximation. In the following, we make two: (i) In the degenerate limit, we have

$$\frac{\mu}{k_B T} \approx \frac{\varepsilon_F}{k_B T} \gg 1, \tag{8}$$

so that we can extend the integration in the last term of Eq. 7 to infinity; note that the integrand of this term decreases rapidly with increasing x—this is why the upper limit of that integral is irrelevant. We cannot do the same with the first term, for example. We obtain:

$$I(\mu) = \int_0^\mu dz \, \varphi(z) + k_B T \int_0^\infty dx \, \frac{1}{e^x + 1} \left[\varphi(\mu + k_B T x) - \varphi(\mu - k_B T x) \right]. \tag{9}$$

(ii) As a second approximation, we expand $\varphi(\mu \pm k_B T x)$ in Taylor series about $k_B T x = 0$, again for the reason that the integrand decreases exponentially with increasing x as well as that $k_B T \ll \mu$:

$$\varphi(\mu + k_B T x) - \varphi(\mu - k_B T x) \approx \varphi(\mu) + k_B T x \frac{d\varphi(\mu)}{d\mu} + \ ... \ - \varphi(\mu) + k_B T x \frac{d\varphi(\mu)}{d\mu} - \ ... \ = 2 k_B T x \frac{d\varphi(\mu)}{d\mu} + \ ... \tag{10}$$

Substituting to our integral, Eq. 11, we get

$$I(\mu) = \int_0^\mu dz \, \varphi(z) + 2 (k_B T)^2 \frac{d\varphi(\mu)}{d\mu} \int_0^\infty dx \, \frac{x}{e^x + 1}. \tag{11}$$

The second integral on the right can be evaluated by expanding $(e^x+1)^{-1}$ into a geometric series (see Appendix 2 to IV.10.):

$$\int_0^\infty dx\,\frac{x^{p-1}}{e^x+1} = \left(1-\frac{1}{2^{p-1}}\right)\zeta(p)\Gamma(p). \tag{12}$$

Then, for $p=2$,

$$\int_0^\infty dx\,\frac{x}{e^x+1} = \frac{1}{2}\zeta(2)\Gamma(2) = \frac{1}{2}\times\frac{\pi^2}{6}\times 1 = \frac{\pi^2}{12}. \tag{13}$$

Using this result, we finally obtain for the low temperature ($T \ll T_F$) expansion,

$$I(\mu) \approx \int_0^\mu dz\,\varphi(z) + \frac{\pi^2}{6}(k_BT)^2\frac{d\varphi(\mu)}{d\mu} + \ \ldots \tag{14}$$

The above result is called Sommerfeld's expansion. It is useful to calculate physical observables of degenerate Fermi gases. For example, if $\varphi(\varepsilon)=\varepsilon^{1/2}$, one needs to substitute $\varphi(z)=z^{1/2}$ and $\varphi(\mu)=\mu^{1/2}$ in the above to obtain

$$I(\mu) = \left|\text{for }\ \varphi(\varepsilon)=\varepsilon^{1/2}\right| = \frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}\frac{(k_BT)^2}{\mu^{1/2}}. \tag{15}$$

This result is used in the class to derive the expression for the chemical potential.