Appendix 2: Averaging over impurity positions
Mesoscopic Physics
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We argued in the lecture that in the expansion
\[ \sum_{i,j=1}^{N_i} e^{-i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} = \sum_{i=j}^{N_i} 1 + \sum_{i \neq j} e^{-i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)}, \] (1)
in which \( \mathbf{q} = \mathbf{k}' - \mathbf{k} \) is the difference of the wave vectors of the scattered and the incident electron wave, and \( \mathbf{R}_i \) are the random positions of impurities, the second term can be ignored. Is it because it is smaller than the first (which equals to \( N_i \), the number of impurities)? The second term contains \( N_i^2 \) (we assume \( N_i \gg 1 \)) numbers of unit magnitude but random phase in the complex plane. The sum therefore averages to zero, if we would average over many impurity configurations. But we have at our disposal only one configuration. The second sum, which contains \( N_i^2 \) terms, is then a number of order \( \sqrt{N_i^2} = N_i \), which is the standard deviation from the average value—exactly like in a random walk. The actual values of the two sums are thus comparable!

There is, however, one important aspect related to the second sum: it is a random function of \( \mathbf{q} \). Typically the smallest phase is
\[ \mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j) \sim \frac{a_{\text{imp}}}{\lambda}, \] (2)
where \( \lambda \) is the wavelength of the scattering electron and \( a_{\text{imp}} \) is the typical distance between impurities. There are two regimes to consider:

(i) \( a_{\text{imp}} \ll \lambda \). In this limit multiple scattering is important and we cannot ignore the second term. Such a situation occurs, for example, in metal alloys. (What is \( \lambda \) in metals?)

(ii) \( a_{\text{imp}} \gg \lambda \). In this limit we can stay within single impurity scattering—this is the case considered in the lecture. The sum, which we can call \( N_i g(\mathbf{q}) \),
\[ N_i g(\mathbf{q}) = \sum_{i \neq j} e^{-i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} \sim O(N_i), \] (3)
is a rapidly oscillating function (positive and negative values equally likely) of \( \mathbf{q} \). If \( q \) changes by \( 1/(4\lambda) \), for example, smallest change of phase is roughly \( a_{\text{imp}}/(4\lambda) \gg 1 \). If we calculate, for example, the scattering probability for a single electron at state \( k \), we get
\[ \frac{1}{\tau_k} = \sum_{k'} W_{k,k'} \] (4)
\[ = \frac{2\pi}{\hbar} \sum_{k'} |\langle k' | V^{\text{imp}}(\mathbf{r}) | k \rangle|^2 \delta(\epsilon_k - \epsilon_{k'}) \] (5)
\[ = \frac{2\pi}{\hbar} N_i \sum_{k'} |V(\mathbf{q})|^2 (1 + S(\mathbf{q})) \delta(\epsilon_k - \epsilon_{k'}). \] (6)

Here \( V(\mathbf{q}) \) is the Fourier transform of the single-impurity potential. If this potential is a smooth function of \( \mathbf{q} \) (or, for that case, \( k' \)), on the scale of \( 1/\lambda \), the product \( |V(\mathbf{q})|^2 g(\mathbf{q}) \) is widely oscillating on this scale. Summation over \( k \) gives then vanishingly small contribution compared to the contribution from summing \( |V(\mathbf{q})|^2 \) only. Since we always sum over wavevectors to get interesting physical properties (such as conductivity), the above example shows that we can safely ignore the contribution from the sum \( N_i g(\mathbf{q}) \) in general.