Some elements of classical gravitation theory

1 Tensors

Modern physics is based on the fundamental concept of symmetry. Within quantum field theory the relevant symmetries can be of quite different nature, e.g. space and time symmetries, gauge symmetries, conformal symmetry, discrete symmetries etc. We are interested here exclusively in space-time symmetries respectively the invariance of physics under the change of inertial systems, i.e. invariance under Lorentz transformations.

The concept of Lorentz tensors can best be introduced in analogy to angular momentum:

Coordinate transformations in 3-dimensions

$$r'_j = \Lambda_{ji} r_i \tag{1}$$

which leave the absolute value invariant (note Einstein summation convention)

$$r'_j r'_j = r_j r_j \tag{2}$$

are a symmetry group of normal 3-dim space. One can introduce quantities with well defined properties under such transformations, e.g. spherical harmonics

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}} \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{\mathrm{i}\phi} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}}\ \cos\theta \\ Y_{1(-1)} &= \sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{-\mathrm{i}\phi} \end{aligned} \tag{3}$$

Under a rotation around the z-axes by $\Delta\phi$ each Y_{lm} transforms with a well defined factor $\exp(\mathrm{i}m\Delta\phi)$. Furthermore, the spherical harmonics form a basis, i.e., every function can be expanded in Y_{lm}.

$$f(r,\theta,\phi) = \sum_{l,m} c_{lm}(r)\ Y_{lm}(\theta,\phi) \tag{4}$$

$$c_{lm}(r) \;=\; \int_0^{2\pi}\int_{-1}^{1} Y^*_{lm}(\theta,\phi)\; f(r,\theta,\phi)\; d\cos\theta\; d\phi \tag{5}$$

For Lorentz-transformations within the special theory of relativity (SRT), which are applicable in the general theory of relativity only to a local (i.e. of infinitesimal extension), flat coordinate system

$$dx'_\mu = \Lambda_\mu{}^\nu dx_\nu \tag{6}$$

the invariant quantity is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{7}$$

with the flat metric

$$\left(\eta_{\mu\nu}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

For the SRT Eq.(7) is not only valid for infinitesimal dx.

Again one can expand all quantities into objects with well-defined transformation properties:
The analog to Y_{00} are Lorentz scalars, i.e. a rank-0 tensors,
the analog to the Y_{1m} are the components of a Lorentz vector, i.e. a rank-1 tensor, (three in 3 dimensions)
the analog to Y_{2m} are symmetric, traceless rank two tensors (of which there are five in 3 dimensions)
etc.
Just as the Y_{lm} have well-behaved transformation properties under rotations, so have tensors. A rank-n tensor transforms with n Lorentz matrices Λ.

$$\begin{aligned} a &\rightarrow a \\ a^\mu &\rightarrow \Lambda^\mu{}_{\mu'} a^{\mu'} \\ a^{\mu\nu} &\rightarrow \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} a^{\mu'\nu'} \\ a^{\lambda\mu\nu} &\rightarrow \Lambda^\lambda_{\lambda'}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} a^{\lambda'\mu'\nu'} \\ &\ldots \end{aligned} \tag{9}$$

Typically $\Lambda^\mu{}_{\mu'}$ is read off from the transformation properties of the coordinate vector x^μ.

For a non-local coordinate system $(\tilde{x})$ with non-trivial curvature (ART) one imposes that ds^2, which is the central physical quantity, has to agree with the ds^2 as determined in a local, flat coordinate system. As the latter is known from SRT this requirement determines the space-time dependent metric $g_{\mu\nu}(x)$ of ART.

$$ds^2 = g_{\mu\nu}(\tilde{x})\; d\tilde{x}^\mu d\tilde{x}^\nu \;\;=\;\; \eta_{\mu\nu}\; dx^\mu dx^\nu \tag{10}$$

A tensor within the framework of general relativity is defined in analogy to Eq.(9) as an object transforming as

$$\begin{aligned} a &\;\rightarrow\; a \\ a^\mu &\;\rightarrow\; B^\mu_{\;\;\mu'} a^{\mu'} \\ a^{\mu\nu} &\;\rightarrow\; B^\mu_{\;\;\mu'} B^\nu_{\;\;\nu'} a^{\mu'\nu'} \\ a^{\lambda\mu\nu} &\;\rightarrow\; B^\lambda_{\lambda'} B^\mu_{\;\;\mu'} B^\nu_{\;\;\nu'} a^{\lambda'\mu'\nu'} \\ &\;\;\;\;\;\ldots \end{aligned} \tag{11}$$

where $B^\mu_{\;\;\mu'}$ can be read off from

$$\begin{aligned} x^\mu &\;\rightarrow\; B^\mu_{\;\;\mu'} x^{\mu'} \;=\; \tilde{x}^\mu \\ B^\mu_{\;\;\mu'} &\;=\; \frac{\partial \tilde{x}^\mu}{\partial x^{\mu'}} \end{aligned} \tag{12}$$

The inverse transformation is

$$\begin{aligned} \overline{B}^\nu_{\;\;\mu} &\;=\; \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \\ \overline{B}^\nu_{\;\;\mu}\; B^\mu_{\;\;\lambda} &\;=\; \delta^\nu_\lambda \end{aligned} \tag{13}$$

To avoid confusion it is practical to call tensors in the sense of SRT 'Lorentz tensors' and those in the sense of ART 'Riemann tensors'.

2 Covariant derivative

In ART as in SRT one aims at writing all physical equations in terms of Tensors, i.e. objects which transform under coordinate transformations according to Eq.(11). This, however, is not trivial to achieve. A typical expression is the derivative of some vector field $A^\beta(x)$, which transform according to:

$$\frac{\partial \tilde{A}^\beta[\tilde{x}(x)]}{\partial \tilde{x}^\alpha} \;\;=\;\; \frac{1}{\partial \tilde{x}^\alpha}\left(B^\beta_{\;\;\mu}[\tilde{x}(x)] A^\mu[\tilde{x}(x)]\right)$$

$$= \; B^{\beta}{}_{\mu}[\tilde{x}(x)] \; \frac{\partial x^{\nu}}{\partial \tilde{x}^{\alpha}} \; \frac{\partial A^{\mu}(x)}{\partial x^{\nu}} + \frac{\partial B^{\beta}{}_{\mu}[\tilde{x}(x)]}{\partial \tilde{x}^{\alpha}} \; A^{\mu}(x)$$

$$= \; B^{\beta}{}_{\mu}(x) \; \overline{B}^{\nu}{}_{\alpha}(x) \; \frac{\partial A^{\mu}(x)}{\partial x^{\nu}} + \frac{\partial B^{\beta}{}_{\mu}[\tilde{x}(x)]}{\partial \tilde{x}^{\alpha}} \; A^{\mu}(x) \qquad (14)$$

which deviates from Eq.(11). Thus the simple derivative does not lead to well defined Lorentz tensors in ART. To correct the additional term one defines the covariant derivative by

$$\tilde{A}^{\beta}{}_{||\alpha} \; := \; \frac{\partial \tilde{A}^{\beta}}{\partial \tilde{x}^{\alpha}} \; + \; \Gamma^{\beta}_{\alpha\lambda}(\tilde{x}) \; \tilde{A}^{\lambda} \; = \; \tilde{A}^{\beta}{}_{|\alpha} \; + \; \Gamma^{\beta}_{\alpha\lambda}(\tilde{x}) \; \tilde{A}^{\lambda} \qquad (15)$$

with the **affine connection** (also called simply 'connection' and in German 'Christoffel-Symbol')

$$\Gamma^{\beta}_{\alpha\lambda} \; = \; \frac{g^{\beta\nu}}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\lambda}} \; + \; \frac{\partial g_{\lambda\nu}}{\partial x^{\beta}} \; - \; \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right) \qquad (16)$$

One can show (see section 3) that

$$\tilde{A}^{\beta}{}_{||\alpha}[\tilde{x}(x)] \; = \; B^{\beta}{}_{\beta'}(x) \; \overline{B}^{\alpha'}{}_{\alpha}(x) \; A^{\beta'}{}_{||\alpha'}(x) \qquad (17)$$

i.e. the covariant derivative leads to tensors.
To understand the physical meaning of the affine connection it is best to consider a freely falling system. In this system and for infinitesimal distances (for finite distances one experiences tidal forces) the equation of motion for any point mass at coordinate ξ^{α} is simply

$$\frac{d^2\xi^{\alpha}}{d\tau^2} \; = \; 0 \qquad (18)$$

with the eigentime (c=1)

$$d\tau^2 \; = \; \eta_{\alpha\beta} d\xi^{\alpha} \; d\xi^{\beta} \qquad (19)$$

This is the free equation of motion within SRT, $d\tau^2 = (1-\beta^2)dt^2$. For the eigentime one finds for example that

$$\tau \; = \; \int_{t_1}^{t_2} dt \; \sqrt{1-\beta^2(t)} \qquad (20)$$

is the eigentime of a clock moved with $\beta(t)$ within the inertial system with time t (time dilatation). The physical line element is

$$ds^2 \; = \; \eta_{\alpha\beta} d\xi^{\alpha} d\xi^{\beta} \; = \; g_{\mu\nu} dx^{\mu} dx^{\nu} \qquad (21)$$

Eq.(18) implies

$$0 = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$0 = \frac{d^2x^\kappa}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\kappa}{\partial\xi^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{22}$$

after contraction with $(\partial x^\kappa/\partial\xi^\alpha)$. A straight-forward, though lengthy calculation shows (see section 3)

$$\Gamma^\kappa_{\mu\nu} = \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\kappa}{\partial\xi^\alpha} \tag{23}$$

$$0 = \frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{24}$$

Thus the affine connection parameterizes all effects of curvature, respectively gravitation. Eq. (24) is the famous equation of free geodetic motion which can also be derived different ways and applies in the most general case.

3 A calculation

In this section we will show that the affine connection from Eq. (24) can also be written in the following form:

$$\Gamma^\kappa_{\mu\nu} = \frac{g^{\kappa\lambda}}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) \tag{25}$$

To prove this statement we simply insert the definition of the metric in terms of some locally flat coordinate system with coordinates ξ^α

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu} \tag{26}$$

into Eq. (25).

$$\begin{aligned}\Gamma^\kappa_{\mu\nu} &= \frac{g^{\kappa\lambda}\eta_{\alpha\beta}}{2}\left(\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial\xi^\beta}{\partial x^\lambda} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu} + \frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\lambda}\right.\\ &\quad \left. + \frac{\partial\xi^\alpha}{\partial x^\nu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\mu} - \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\lambda}\frac{\partial\xi^\beta}{\partial x^\nu} - \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\nu\partial x^\lambda}\right)\\ &= g^{\kappa\lambda}\eta_{\gamma\beta}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial\xi^\beta}{\partial x^\lambda}\,\delta^\gamma_\alpha\end{aligned}$$

$$
\begin{aligned}
&= g^{\kappa\lambda}\eta_{\gamma\beta}\ \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^\beta}{\partial x^\lambda}\ \frac{\partial \xi^\gamma}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha} \\
&= g^{\kappa\lambda}\left(\eta_{\gamma\beta}\frac{\partial \xi^\beta}{\partial x^\lambda}\frac{\partial \xi^\gamma}{\partial x^\rho}\right)\ \frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} \\
&= \frac{\partial x^\kappa}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} \qquad (27)
\end{aligned}
$$

Next we prove Eq.(18). With the additional term Eq.(**??**) reads

$$
\begin{aligned}
A^\beta_{||\alpha}[\tilde{x}(x)] &= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ \frac{\partial A^\mu(x)}{\partial x^\nu} + \frac{\partial B^\beta{}_\mu[\tilde{x}(x)]}{\partial \tilde{x}^\alpha}\ A^\mu(x) \\
&+ \frac{\partial \tilde{x}^\beta}{\partial \xi^\gamma}\frac{\partial^2 \xi^\gamma}{\partial \tilde{x}^\alpha \partial \tilde{x}^\lambda} B^\lambda{}_{\beta'}\ A^{\beta'}(x) \\
&= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ \frac{\partial A^\mu(x)}{\partial x^\nu} + \frac{\partial B^\beta{}_\mu[\tilde{x}(x)]}{\partial \tilde{x}^\alpha}\ A^\mu(x) \\
&+ \frac{\partial \tilde{x}^\beta}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\gamma}\frac{\partial}{\partial \tilde{x}^\alpha}\left(\frac{\partial \xi^\gamma}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial \tilde{x}^\lambda}\right) B^\lambda{}_{\beta'}\ A^{\beta'}(x) \\
&= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ \frac{\partial A^\mu(x)}{\partial x^\nu} + \frac{\partial B^\beta{}_\mu[\tilde{x}(x)]}{\partial \tilde{x}^\alpha}\ A^\mu(x) \\
&+ \left(\Gamma^\rho_{\tau\sigma}\frac{\partial \tilde{x}^\beta}{\partial x^\rho}\frac{\partial x^\tau}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\lambda}\ +\ \frac{\partial \tilde{x}^\beta}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial \tilde{x}^\lambda \partial \tilde{x}^\alpha}\right) B^\lambda{}_{\beta'}\ A^{\beta'}(x) \\
&= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ \frac{\partial A^\mu(x)}{\partial x^\nu} + \frac{\partial B^\beta{}_\mu[\tilde{x}(x)]}{\partial \tilde{x}^\alpha}\ A^\mu(x) \\
&+ B^\beta{}_\rho \overline{B}^\tau{}_\alpha B^\lambda{}_{\beta'} \overline{B}^\sigma{}_\lambda \Gamma^\rho_{\tau\sigma} A^{\beta'}\ +\ B^\beta{}_\rho \frac{\partial \overline{B}\,{}^\rho{}_\lambda}{\partial \tilde{x}^\alpha} B^\lambda{}_{\beta'}\ A^{\beta'} \\
&= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ \frac{\partial A^\mu(x)}{\partial x^\nu}\ +\ B^\beta{}_\rho(x)\ \overline{B}^\tau{}_\alpha(x)\ \Gamma^\rho_{\tau\beta'}(x) A^{\beta'} \\
&= B^\beta{}_\mu(x)\ \overline{B}^\nu{}_\alpha(x)\ A^\mu_{||\nu}(x) \qquad (28)
\end{aligned}
$$

where we have used in the last step the identities

$$
\begin{aligned}
B^\beta{}_\rho(x)\ \overline{B}^\rho{}_\lambda(x) &= \delta^\rho_\lambda \qquad (29) \\
\frac{\partial}{\partial \tilde{x}^\alpha}\left(B^\beta{}_\rho(x)\ \overline{B}^\rho{}_\lambda(x)\right) &= 0 \\
\Rightarrow \qquad \frac{\partial \overline{B}^\rho{}_\lambda}{\partial \tilde{x}^\alpha} B^\beta{}_\rho(x) &= -\overline{B}^\rho{}_\lambda(x)\frac{\partial B^\beta{}_\rho}{\partial \tilde{x}^\alpha} \\
\Rightarrow \qquad \frac{\partial \overline{B}^\rho{}_\lambda}{\partial \tilde{x}^\alpha} B^\beta{}_\rho B^\lambda_{\beta'} &= -\frac{\partial B^\beta{}_{\beta'}}{\partial \tilde{x}^\alpha} \qquad (30)
\end{aligned}
$$

4 Parallel transport

As we have seen, if $A^\alpha(x)$ is a covariant vector,

$$dA^\alpha(x) \;=\; A^\alpha_{\;|\beta}(x)dx^\beta \;=\; A^\alpha(x+dx) \;-\; A^\alpha(x) \tag{31}$$

is not, but only

$$DA^\alpha(x) \;=\; A^\alpha(x+dx) \;-\; A^\alpha(x) \;-\; \Gamma^\alpha_{\beta\gamma}A^\beta dx^\gamma \tag{32}$$

The additional term is in general not invariant under parallel transport, see figure 1.

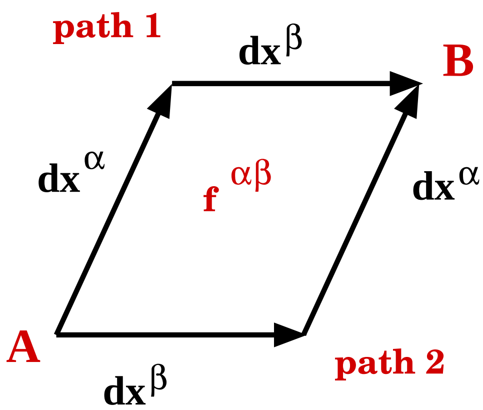

Figure 1: Parallel transport in curved space

One gets

$$\int_{A,path\ 1}^{B} \Gamma^\lambda_{\rho\sigma}A_\lambda dx^\sigma \;-\; \int_{A,path\ 2}^{B} \Gamma^\lambda_{\rho\sigma}A_\lambda dx^\sigma \;=\; \oint_{path\ 1+2} \Gamma^\lambda_{\rho\sigma}A_\lambda dx^\sigma$$
$$=: \; -\frac{1}{2}R^\lambda_{\;\rho\alpha\beta}A_\lambda df^{\alpha\beta} \tag{33}$$

which defines the **curvature tensor** $R^\lambda_{\;\rho\alpha\beta}(x)$. Inserting our expressions for the covariant derivative one can express $R^\lambda_{\;\rho\alpha\beta}(x)$ in terms of affine connections and the metric.

$$\begin{aligned} A_{\sigma||\alpha||\beta} \;-\; A_{\sigma||\beta||\alpha} &= \left(A_{\sigma|\alpha} - \Gamma^\lambda_{\sigma\alpha}A_\lambda\right)_{||\beta} \;-\; \left(A_{\lambda|\beta} - \Gamma^\lambda_{\sigma\beta}A_\lambda\right)_{||\alpha} \\ &= -\left(\Gamma^\lambda_{\sigma\alpha|\beta} \;-\; \Gamma^\lambda_{\sigma\beta|\alpha} \;-\; \Gamma^\rho_{\sigma\beta}\Gamma^\lambda_{\rho\alpha} \;+\; \Gamma^\rho_{\sigma\alpha}\Gamma^\lambda_{\rho\beta}\right)A_\lambda \\ &=: \; -R^\lambda_{\;\sigma\alpha\beta}A_\lambda \end{aligned} \tag{34}$$

The definition of $R^{\lambda}{}_{\sigma\alpha\beta}$ can be rewritten with some effort in the standard form:

$$\begin{aligned} R^{\lambda}{}_{\sigma\alpha\beta} &= \Gamma^{\lambda}_{\sigma\alpha|\beta} - \Gamma^{\lambda}_{\sigma\beta|\alpha} - \Gamma^{\rho}_{\sigma\beta}\Gamma^{\lambda}_{\rho\alpha} + \Gamma^{\rho}_{\sigma\alpha}\Gamma^{\lambda}_{\rho\beta} \\ R_{\lambda\sigma\alpha\beta} &= \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\alpha}}{\partial x^{\sigma}\partial x^{\beta}} + \frac{\partial^2 g_{\sigma\beta}}{\partial x^{\lambda}\partial x^{\alpha}} - \frac{\partial^2 g_{\sigma\alpha}}{\partial x^{\lambda}\partial x^{\beta}} - \frac{\partial^2 g_{\lambda\beta}}{\partial x^{\sigma}\partial x^{\alpha}}\right) \\ &+ g_{\rho\tau}\left(\Gamma^{\rho}_{\alpha\lambda}\Gamma^{\tau}_{\sigma\beta} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\tau}_{\sigma\alpha}\right) \end{aligned} \tag{35}$$

The Einstein-equation, which we will drive in the following section, is an equation between rank-2 tensors. Therefore, we have to build a rank two tensor of the rank-4 curvature tensor, which can only be done by contracting two of the four indices, i.e. by calculating:

$$g^{\lambda\sigma}R_{\lambda\sigma\alpha\beta} \quad \text{or} \quad g^{\lambda\alpha}R_{\lambda\sigma\alpha\beta} \quad \text{or} \quad g^{\sigma\beta}R_{\lambda\sigma\alpha\beta} \quad \text{or} \quad etc.$$

However, $R_{\lambda\sigma\alpha\beta}$ is antisymmetric both under the exchange of the first two and the last two indices. Therefore, there exists only one contraction. All others are either equivalent or zero. This contraction defines the **Ricci tensor**.

$$R_{\alpha\beta} = g^{\lambda\sigma}R_{\lambda\sigma\alpha\beta} \tag{36}$$

We will also need the scalar curvature, defined by

$$R = g^{\alpha\beta}R_{\alpha\beta} \tag{37}$$

Finally we will need an identity fulfilled by the curvature tensor, namely the **Bianchi Identity**.

$$R_{\lambda\sigma\alpha\beta||\gamma} + R_{\lambda\sigma\beta\gamma||\alpha} + R_{\lambda\sigma\gamma\alpha||\beta} = 0 \tag{38}$$

Its proof is actually rather easy if one uses the fact that we are free to choose the coordinate frame in which we perform the calculation. If the given combination is zero in one frame it stays zero under coordinate transformations. Therefore we choose locally flat coordinates ξ^{α} such that all affine connectors vanish.

$$\begin{aligned} R_{\lambda\sigma\alpha\beta||\gamma} &= \frac{1}{2}\frac{\partial}{\partial\xi^{\gamma}}\left(\frac{\partial^2 g_{\lambda\alpha}}{\partial\xi^{\sigma}\partial\xi^{\beta}} + \frac{\partial^2 g_{\sigma\beta}}{\partial\xi^{\lambda}\partial\xi^{\alpha}}\right. \\ &\left. - \frac{\partial^2 g_{\sigma\alpha}}{\partial\xi^{\lambda}\partial\xi^{\beta}} - \frac{\partial^2 g_{\lambda\beta}}{\partial\xi^{\sigma}\partial\xi^{\alpha}}\right) \end{aligned} \tag{39}$$

Inserting Eq.(39) into Eq.(38) one simply finds that all terms cancel.

5 The Einstein equation

Based on everything we have said so far, the derivation of the Einstein equation is rather straight-forward. The basic physical facts which have to be described by the Einstein equation are the following:
1.) Inertial mass and gravitational mass are equal, e.g. in Newtonian mechanics

$$m(\text{inertial})\frac{\partial^2\vec{x}}{\partial t^2} = -G\frac{Mm(\text{grav})}{\left|\vec{x}-\vec{R}\right|^3}\,(\vec{x}-\vec{R}) \tag{40}$$

$$m(\text{inertial}) = m(\text{grav}) \tag{41}$$

Therefore, the masses cancel and e.g. the same orbit can be followed by a planet of larger or smaller mass. This independence of the trajectory of the properties of the test particle, i.e. its mass allows to attribute the effect of gravitation to a general, i.e. test particle independent, curvature of space-time.
2.) All forms of energy gravitate. This is e.g. tested experimentally by measuring the red-shift of photons (which are pure electro-magnetic field energy) in the gravitational field of the earth.
3.) The tensor which contains the energy is the energy momentum tensor $T_{\mu\nu}$.

Thus the Einstein equation must have the form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{42}$$

$$\text{curbvature of space} - \text{time} = \text{energy} - \text{momentum tensor}$$

where $G_{\mu\nu}$ must fulfil:
i.) it must be a rank-2 tensor which is determined purely by the curvature of space time.
ii.) it must be symmetric, because $T_{\mu\nu}$ is symmetric.
iii.) Energy momentum conservation requires in curved space-time

$$T^{\mu\nu}{}_{||\nu} = 0 \tag{43}$$

Eq.(42) thus implies the requirement:

$$G^{\mu\nu}{}_{||\nu} = 0 \tag{44}$$

The factor 8π is purely conventional and G is the gravitational constant.

The only two symmetric rank-2 tensors from which we can build $G_{\mu\nu}$ are the Ricci tensor and $g_{\mu\nu}$. However the latter must be multiplied with R, as it has to vanish when $T_{\mu\nu}$ vanishes, i.e. when the curvature vanishes. Higher powers of R are not possible as the dimensions of all terms must be the same. (R has the dimension (length)^2. Thus the most general ansatz reads

$$G_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} \tag{45}$$

Contracting the Biancchi identity

$$\begin{aligned} 0 &= R^{\rho}{}_{\mu\sigma\nu||\lambda} + R^{\rho}{}_{\mu\nu\lambda||\sigma} + R^{\rho}{}_{\mu\lambda\sigma||\nu} \\ &= R^{\rho}{}_{\mu\sigma\nu||\lambda} - R^{\rho}{}_{\mu\lambda\nu||\sigma} - R^{\rho}{}_{\mu\sigma\lambda||\nu} \end{aligned} \tag{46}$$

with $\delta^{\sigma}{}_{\rho}\, g^{\mu\nu}$ results in

$$\begin{aligned} 0 &= R_{||\lambda} - R^{\rho}{}_{\lambda||\rho} - R^{\nu}{}_{\lambda||\nu} \\ 2R^{\nu}{}_{\mu||\nu} &= R_{||\mu} \end{aligned} \tag{47}$$

Thus we get

$$\begin{aligned} G^{\nu}{}_{\mu||\nu} &= \frac{a}{2}R_{||\mu} + bR_{||\mu} \stackrel{!}{=} 0 \\ b &= -\frac{a}{2} \end{aligned} \tag{48}$$

Finally a is fixed by considering the non-relativistic limit with only a weak gravitational field. This gives $a = -1$, which ends our derivation of Einsteins equation.

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \tag{49}$$

Often the Einstein equation is written in a slightly different form. Subtracting from each side of the equation the trace of that side divided by two and multiplied with $g_{\mu\nu}$ we get for the left hand side

$$G_{\mu\nu} - \frac{1}{2}G^{\mu}{}_{\mu}g_{\mu\nu} = -R_{\mu\nu} + \frac{R}{2}g_{\mu\nu} - \frac{1}{2}(-R + 2R) = -R_{\mu\nu} \tag{50}$$

and with the definition

$$T = g^{\mu\nu}T_{\mu\nu} \tag{51}$$

the equation

$$\boxed{R_{\mu\nu} = -\,8\pi G\Big(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\Big)} \tag{52}$$