2. Observed mass range of BHs. Static BHs. Schwarzschild metric.

2.1. How to find Black Holes.

If radiation cannot escape from a Black Hole (BH), how can it then be discovered? Usually the BH is surrounded by an accretion disk of matter which moves on Kepler orbits around the centre Since matter (from stars or interstellar gas) forms near the BH a hot plasma we expect the presence of magnetic fields. This in turn leads to friction effects. The system loses angular momentum and the matter spirals slowly to the horizon of the BH. In its final (stable) orbit the potential energy of the strong gravitational field has been converted into kinetic energy of the particles in the plasma which easily reaches temperatures of some 10^7 Kelvin. The hot plasma is a source of an intensive radiation mainly in the X-ray region where a considerable fraction of mc^2 is radiated away. This intensive X-ray radiation is considered to be a fingerprint of BHs or neutron stars (NSs). There exist a correlation between the state of the innermost hot plasma and the accretion rate which is the mass per unit of time swallowed by the BH.

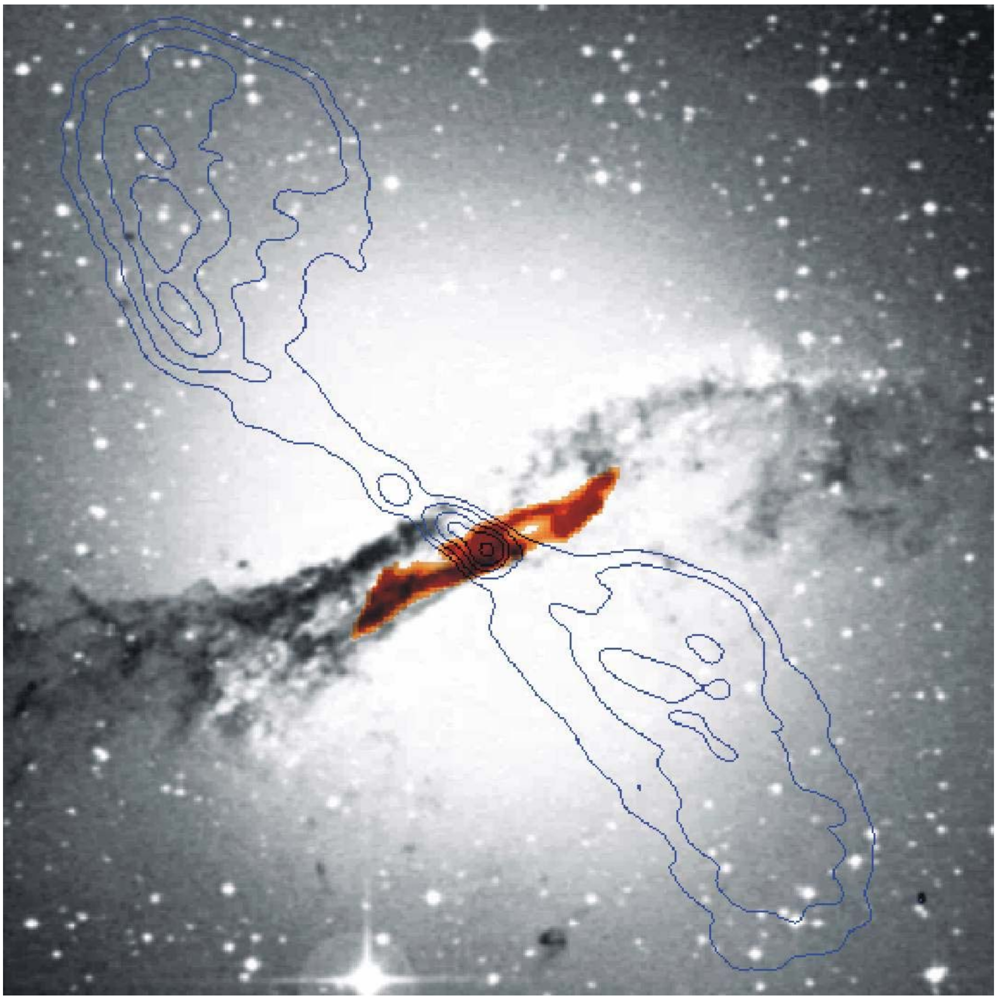

Fig. 2.1. The active galaxy Cen A. The picture combines images from radio and optical exposures.. The central region harbors a BH with about 10^7 solar masses. Also seen is a giant jet of relativistic particles.
Credit: X-ray - NASA, CXC, R.Kraft (CfA), et al.; Radio - NSF, VLA, M.Hardcastle (U Hertfordshire) et al.; Optical - ESO, M.Rejkuba (ESO-Garching) et al.

BHs are found in three mass ranges:

1) Stellar BHs have a few solar masses and are formed as final state of massive stars with $M > 15\ M_{solar}$. This last transition into a BH is a catastrophic effect sometimes called a hypernova with an energy burst of about 10^{48} Joule (100 times that of a normal supernova). It is detected as a Gamma ray burst (GRB) lasting a few seconds till some minutes. The Swift-satellite is specially constructed and launched to detect these fast events. Good mass estimates are only possible in binary systems when a BH and a normal star are orbiting around their common centre using Kepler's 3rd law..
2) Super massive BHs are found in the center of practically all larger galaxies. Their striking feature is the strong X- radiation but also radiation in optical and radio frequencies. Furthermore the radial velocities of the stars increase strongly near the center of the galaxy. The BH masses can be estimated from the dynamics of the stars near the BH and range from 10^6 to some 10^9 solar masses. In spiral galaxies with an inner spherical bulge the BH mass is roughly $10^{-3} \cdot M_{bulge}$.
3) BH in the intermediate mass range should have masses from $M = 10^3 - 10^5$ solar masses. This range is less well established. There are some conjectures of the formation of BHs in very dense stellar clusters like the inner parts of globular clusters (GCs) or dwarf galaxies. GCs lack of interstellar matter and therefore we do not expect radiation from accretion. So we still wait for convincing examples.

2.2. Metric and equivalence principle.

We know well that all masses fell equally fast, a consequence of the equality of gravitating and accelerating mass which led Einstein to the conjecture that masses change the geometry of space. Thus in regions curved by a big mass all test masses move in equal orbits when started under equal conditions. Every free-falling or freely orbiting "spacelab" is a force free local reference system or inertial system governed by Minkowski metric. In the presence of matter Einstein's equations describe the relation between the distribution of matter and the local geometry. This will be discussed in section 2.5.

The distance ds of two neighboring points on a curved surface or in curved space is in general given by

$$ds^2 = \sum g_{ik} (dx^i)(dx^k)$$

The coefficients g_{ik} are functions of space coordinates that is the metric is a local property of spacetime and changes from point to point. A good 2-dimensional example is the surface of the earth

$$ds^2 = r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)$$

Flat spacetime is described by the Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \sum \eta_{ln} (dx^l)(dx^n) \qquad (2.1)$$

which is still a good approximation far away from gravitating masses. In special relativity (2.1) is a global property of spacetime and the metric tensor has a simple form

$$(\eta_{ik}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{2.2}$$

Usually the metric g_{ik} is given by the symmetry of the problem which we have to solve. If the metric is known one may use Einstein's equations to connect metric with mass. As already mentioned a well known example is a spacecraft orbiting the earth (e.g. the International Space Lab). If that spacecraft would use its engine to achieve a constant acceleration as big as the gravitational acceleration on the earth the physics inside the spacecraft would be the same as that in a terrestrial lab. This is just an example of the more far reaching ***equivalence principle*** which is always locally obeyed. It is basically the consequence of the equality of gravitating and accelerating mass mentioned above which is experimentally very well established.

2.3. Geodesics

The "shortest distance" between two points on a curved surface or in a curved 3-space is a geodesic line. We may use the metric expression $ds^2 = \sum g_{ik}(dx^i)(dx^k)$ to determine a general expression for a geodesic by looking for an extremum

$$\partial \int \sqrt{g_{ik}(x)\dot{x}^i \dot{x}^k}\, d\lambda = \partial \int L' d\lambda = 0 \tag{2.3}$$

However, it is easier to consider

$$\partial \int g_{ik}(x)\dot{x}^i \dot{x}^k d\lambda = \partial \int 2L d\lambda = 0 \tag{2.4}$$

This latter expression (2.4) avoids the square root and is equivalent to (2.3). Here λ is an arbitrary parameter (sometimes called affine parameter) on the world line. We can take $d\lambda = ds$. The Euler equations of this variational problem are

$$\frac{\partial L}{\partial x^i} = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^i} \tag{2.5}$$

We choose $\frac{dL}{d\lambda} = 0$ and find after some lengthy operations the geodesic equation

$$\ddot{x}^n + \Gamma^n_{kl}\dot{x}^k \dot{x}^l = 0 \tag{2.6}$$

where the Christoffel symbols Γ^n_{kl} are obtained from the g_{kl} in the following way

$$\Gamma_{mnl} = \frac{1}{2}(g_{mn,l} + g_{ml,n} - g_{nl,m}) \qquad \text{and} \quad \Gamma^k_{nl} = g^{km}\Gamma_{mnl} \tag{2.7}$$

where

$$g_{kl,n} = \frac{\partial}{\partial x^n} g_{kl} \quad \text{and} \quad g_{mk} g^{kl} = \delta^l_m .$$

To give a simple application we work out the ***equation of motion in Newtonian gravity***. The gravitational potential should be a small perturbation of the metric (2.2)

$$g_{kl} = \eta_{kl} + 2\psi_{kl}(x) \tag{2.8}$$

with

$$\lfloor \psi_{kl} \rfloor << 1 \tag{2.9}$$

The 4-velocity

$$\dot{x}^k = (1, \upsilon^k) \tag{2.10}$$

is given in units of c. In the limits of Newtonian mechanics we have always $\upsilon^k << 1$ (motion of planets) and we may write

$$\dot{x}^k \cong (1,0) \text{ and } ds = c\,dt\sqrt{1-\upsilon^2} \approx cdt \tag{2.11}$$

In the same approximation we have

$$\frac{d^2 x^k}{ds^2} \approx \frac{d^2 s}{c^2 dt^2} \tag{2.12}$$

and

$$\frac{d^2 x^k}{c^2 dt^2} + \Gamma^k_{00} = 0 \tag{2.13}$$

Note that $\upsilon^0 \cong 1$. For the Christoffel symbol we obtain

$$\Gamma^k_{00} = \eta^{km} \Gamma_{m00} = -\Gamma_{k00} = -\frac{1}{2}\left(\frac{\partial g_{0k}}{\partial x^0} + \frac{\partial g_{0k}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^k} \right) \tag{2.14}$$

Since we only consider <u>static</u> gravitational fields we may drop the derivities with respect to time $\frac{\partial}{\partial x^0}$ which simplifies the expression

$$\Gamma^k_{00} = \frac{1}{2}\frac{\partial g_{00}}{\partial x^k} = \frac{\partial \psi_{00}}{\partial x^k} \tag{2.15}$$

Then the acceleration becomes

$$\frac{1}{c^2}\frac{d^2 x}{(dt)^2} = -\frac{\partial \psi_{00}}{\partial x^2} = -\frac{\partial \psi_{00}}{\partial r} \tag{2.16}$$

Comparing this result with the well known expression from Newtonian gravitation we may write

$$\psi_{00} = -\frac{1}{c^2} U = -\frac{1}{c^2}\frac{GM}{r} \tag{2.17}$$

and finally we have

$$g_{00} \approx 1 + 2\psi_{00} \approx 1 - \frac{2GM}{c^2 r} = 1 - \frac{r_S}{r} \qquad (2.18)$$

2.4. The Schwarzschild metric.

Einstein published the equations of his GR 1915 in the “Sitzungsberichten d. Preuß. Akad. Wissenschaften”. 1916, a few months later Karl Schwarzschild, director of the Astrophysical Institute in Potsdam but this time a soldier on leave of absence from the front in Russia, read Einstein’s paper and immediately started to apply Einstein’s equations to a

Fig. 2.2. Karl Schwarzschild (1873 – 1916) 1906 – 1916 Director of the Astrophysical Observatory Potsdam

spherical mass and found a simple looking solution. To see this we first rewrite the Minkowski metric in spherical coordinates

$$ds^2 = c^2 dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \qquad (2.18)$$

Now Schwarzschild found that the most general metric outside a mass of spherical symmetry has the form

$$ds^2 = c^2 dt^2 f(r) + dr^2 h(r) - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (2.19)$$

In what follows we mostly set c = 1, that is velocities are measured in fractions of c. The functions *f* and *h* are given by

$$f(r) = \left(1 - \frac{r_S}{r}\right) \text{ and } h(r) = -\left(1 - \frac{r_S}{r}\right)^{-1} \qquad (2.20)$$

Indeed, since we consider a static problem the metric must be symmetric to time inversion $(t \to -t)$, that is all components of the metric tensor linear in *dt* must be zero $g_{0k} = 0$. Far away from the spherical mass $(r \to \infty)$ the Minkowski metric should be restored which is the case with the expressions (2.20).

$$f(r \to \infty) = 1 \text{ and } h(r \to \infty) = -1 \qquad (2.21)$$

On the other hand when the mass approaches zero $M \to 0$ then the Schwarzschild radius also vanishes $r_S \to 0$ and

$$f(M \to 0) = 1 \text{ and } h(M \to 0) = -1 \tag{2.22}$$

Again the Minkowski metric is restored. The Schwarzschild metric (c = 1)

$$ds^2 = (1 - \frac{r_S}{r})\, dt^2 - \frac{dr^2}{(1 - \frac{r_S}{r})} - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \tag{2.23}$$

has a singularity in g_{11} for $r = r_S$. Actually this is a coordinate singularity which may be removed after transformation to an appropriate coordinate system. When Einstein received Schwarzschild's paper he is quoted to be said: "*I had not expected that one could formulate the exact solution of the problem in such a simple way*".

A consistent derivation of (2.23) is only possible with Einstein's equations. We will give a derivation in the following subsection.

2.5. Einstein's Equations.

We have introduced r_S within Newtonian approximation. We will now derive r_S and the Schwarzschild metric (2.20) in the frame of GR. The basis of GR is a tensorial equation which looks quite simple

$$G_{ik} = \kappa T_{ik} \tag{2.24}$$

The left side is the Einstein tensor and decribes the local geometry

$$G_{ik} = R_{ik} - \frac{1}{2} R \, g_{ik} \tag{2.25}$$

It is a combination of the Ricci tensor R_{ik} and the Ricci scalar.

$$R = \sum_k g^{ik} R_{ik} \equiv g^{ik} R_{ik} \tag{2.26}$$

The sum in (2.26) runs over equal indices. Following Einstein the sum sign will be dropped. T_{ik} is the energy momentum tensor. It describes the matter and its distribution. κ a constant

$$\kappa = \frac{8\pi G}{c^4} \tag{2.27}$$

In 4-dimensional spacetime (2.24) is a shorthand notation of 10 equations. When the Schwarzschild metric is used only 4 diagonal elements R_{kk} are left. We restrict to the field outside of the mass M where $T_{kk} = R_{kk} = 0$. Note that T_{ik} describes in our case a star built of a classical or quantum gas with the parameters density and pressure. We return to (2.20) and use now the following Ansatz for $f(r)$ and $h(r)$

$$f(r) = \exp 2\alpha(r) \quad \text{and} \quad h(r) = \exp 2\beta(r) \tag{2.28}$$

R_{ik} can be obtained from the Christoffel symbols in the following way

$$R_{ik} = \Gamma^{l}_{ik,l} - \Gamma^{l}_{il,k} + \Gamma^{l}_{ik}\Gamma^{m}_{lm} - \Gamma^{m}_{il}\Gamma^{l}_{km} \tag{2.29}$$

They are calculated directly from the components of the metric tensor

$$\Gamma^{m}_{nl} = \frac{1}{2} g^{mk}(g_{kn,l} + g_{kl,n} - g_{nl,k}) \tag{2.30}$$

with
$$(g_{kl}) = \begin{bmatrix} \exp 2\alpha(r) & 0 & 0 & 0 \\ 0 & -\exp 2\beta(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix} \tag{2.31}$$

The Christoffel symbols thus obtained are listed below

$$\begin{array}{ll} \Gamma^{0}_{01} = \Gamma^{0}_{10} = \alpha' & \Gamma^{1}_{00} = \alpha' \exp 2(\alpha - \beta) \\ \Gamma^{3}_{23} = \cot\theta & \Gamma^{1}_{22} = -r \exp -2\beta \\ \Gamma^{2}_{21} = \Gamma^{2}_{12} = \frac{1}{r} & \Gamma^{2}_{33} = -\sin\theta\cos\theta \\ \Gamma^{3}_{13} = \Gamma^{3}_{13} = \frac{1}{r} & \Gamma^{1}_{11} = \beta' \\ \Gamma^{1}_{33} = -r\sin^2\theta \exp -2\beta & \end{array} \tag{2.32}$$

Note that the Γ^{m}_{kl} are symmetric in two lower indices

$$\Gamma^{m}_{kl} = \Gamma^{m}_{lk}\,. \tag{2.33}$$

All other components in (2.32) vanish $\Gamma^{n}_{kl} = 0$ .The dashed functions are derivatives $\alpha'(r) = \frac{\partial}{\partial r}\alpha(r)$ etc. The vacuum solutions have

$$T_{kl} = 0$$

which implies

$$R_{kl} - \frac{1}{2}R = 0 \qquad \text{and} \qquad R_{kl} = \frac{8\pi G}{c^4}(T_{kl} - \frac{1}{2} g_{kl} T) = 0 \tag{2.34}$$

With the values of (2.32) we can calculate R_{kl} (see 2.29). Only the diagonal elements contribute. Students should try it themselves. As an example we evaluate R_{00} and R_{11}

$$R_{00} = \Gamma^{l}_{00,l} - \Gamma^{l}_{0l,0} + \Gamma^{l}_{00}\Gamma^{m}_{lm} - \Gamma^{m}_{0l}\Gamma^{l}_{0m} \tag{2.35}$$

$$R_{11} = \Gamma^{l}_{11,l} - \Gamma^{l}_{1l,1} + \Gamma^{l}_{11}\Gamma^{m}_{lm} - \Gamma^{m}_{1l}\Gamma^{l}_{1m}$$

Inserting the non vanishing Christoffel Symbols from (2.32) we obtain

$$R_{00} = \left[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right] \cdot \exp 2(\alpha - \beta) = 0 \tag{2.36}$$

and

$$R_{11} = \alpha'^2 + \alpha'' - \alpha'\beta' - \frac{2}{r}\beta' = 0 \tag{2.37}$$

$$R_{22} = -1 + (\alpha' - \beta')\, r \exp(-2\beta) + \exp(-2\beta) = 0 \tag{2.38}$$

$$R_{33} = \sin^{22}\theta \cdot R_{22} = 0 \tag{2.39}$$

The following sum

$$R_{11} \cdot \exp - 2\beta + R_{00} \cdot \exp - 2\alpha = -\frac{2}{r}\exp - 2\beta \cdot (\alpha' + \beta') = 0 \tag{2.40}$$

yields

$$(\alpha + \beta) = const. \tag{2.41}$$

With (2.41) we eliminate α in R_{22} and obtain the following differential equ.

$$-1 + \exp - 2\beta + r\frac{d}{dr}\exp - 2\beta = 0 \tag{2.42}$$

With the general solution

$$\exp - 2\beta = 1 + \frac{b}{r} \tag{2.43}$$

From (2.41) it follows

$$\exp 2\alpha = a\left(1 + \frac{b}{r}\right) \tag{2.44}$$

Where $a > 0$ is an integration constant which can be absorbed in a recalibrated time coordinate $\tilde{t} = \sqrt{a} \cdot t$. From (2.17) and (2.18) follows

$$b = -\frac{2GM}{c^2} = -r_S \tag{2.45}$$

Using Einstein's equations we finally have shown that

$$g_{00} = \left(1 - \frac{r_S}{r}\right) \text{ and } -g_{11} = \left(1 - \frac{r_S}{r}\right)^{-1} \tag{2.46}$$

And thus reproduced the result which Karl Schwarzschild found 1916 a few months before he died from a disease which he contracted during the war at the eastern front in Russia.

2.6. Problems

2.6.1. Show that $R_{kl} = \kappa(T_{kl} - \frac{1}{2} g_{kl} T)$ is just another form of Einstein's equations. Hint: Use (2.26) and $g_{kl} \cdot g^{kn} = \delta_k^n$ and $g_{00} g^{00} = 1$, $g^{11} g_{11} = 1$ ect., therefore $g_{kk} \cdot g^{kk} = 4$.

2.6.2. Consider the surface of a sphere. Its metric is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2$$

Write down the components of the metric tensor.

In order to find the respective geodesic use the following Lagrangian

$$L = \frac{m}{2} \cdot \left[r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\phi}^2 \right]$$

The dotted quantities are derivatives with respect to time. Now use

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k} \right) - \frac{\partial L}{\partial x^k} = 0 \quad \text{with} \quad x^1 = \theta, \ \ x^2 = \phi \ ,$$

Set $\theta = \pi / 2$ and $\dot{\theta} = 0$, you will find a conservation law. Use it to formulate the orbital motion (in the equatorial plane).

For those of you, who have already red section 2.5. problem 2.6.3. is optional not obligatory!!.

[2.6.3. Determine the components R_{00} and R_{11}. of the Ricci-tensor (2.29)

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km}$$

by using the given Christoffel Symbols Γ^n_{kl} of (2.32).]

2.6.4. In the center of our galaxy resides a super massive BH with about $4 \cdot 10^6$ solar masses. Its distance from earth is 8,3 kpc. What is the minimal angular resolution (given in microarc seconds µas) which we need to resolve the event horizon? Assume a non-rotating BH and that we are looking face on the accretion disk of the SMBH. Neglect any enlargement by gravitational lensing (1 pc = $3{,}08 \cdot 10^{13}$ km).

Books and Articles.

Andreas Müller: Schwarze Löcher. Das dunkelste Geheimnis der Gravitation.
http://www.wissenschaft-online.de/astrowissen/downloads/Web-Artikel/SchwarzeLoecher_AMueller2007.pdf
A very good introduction to the astrophysics of BHs. Not always popular but always good to read. The text is to download, again a book of 137 pages for nothing. (The author would be glad to get a response from you or any kind of appreciation!!)

Hubert Goenner: Einführung in die spezielle und allgemeine Relativitätstheorie. Spektrum Akademischer Verlag. Heidelberg, Berlin Oxford 1996
A very voluminous book to big to read but contains many clever examples

Bernard Shutz: A first course in relativity. Cambridge University Press 1985
A modern presentation, very clear and not too long. The author, an American physicist is today director at the Albert Einstein Institut of gravitational physics.

P.A.M. Dirac: General Theory of Relativity. Princeton landmarks in mathematical physics 1975.
This is the shortest introduction to GR I know, only 68 pages, very clear and understandable

Sean M. Carroll: Lecture Notes on G3eneral Relativity. Submitted 3 Dec 1997
http://arxiv.org/abs/gr-qc/9712019
A very good text, every thing nicely explained, very up-to-date. You can download the text of 231 pages under the address given above. It is a very good and useful textbook for nothing!!