3. **Dynamics and test particle motion.**

3.1. X-ray Binaries

Systems with 2 stars orbiting around their common center of mass are relatively abundant. If one of the components is a BH we have a chance to determine its mass. A direct observation of the motion is only possible in systems which are near to earth. However, in all other cases it is at least possible to measure the radial velocities υ_{1r} and υ_{2r} using the ***Doppler Effect***. We obtain two sin-curves. The angle “i” measures the inclination of the orbital plane to the line of sight.

$$\upsilon_{1r} = \upsilon_1 \sin i \cdot \sin \omega t \quad \text{and} \quad \upsilon_{2r} = -\upsilon_2 \sin i \cdot \sin \omega t \tag{3.1}$$

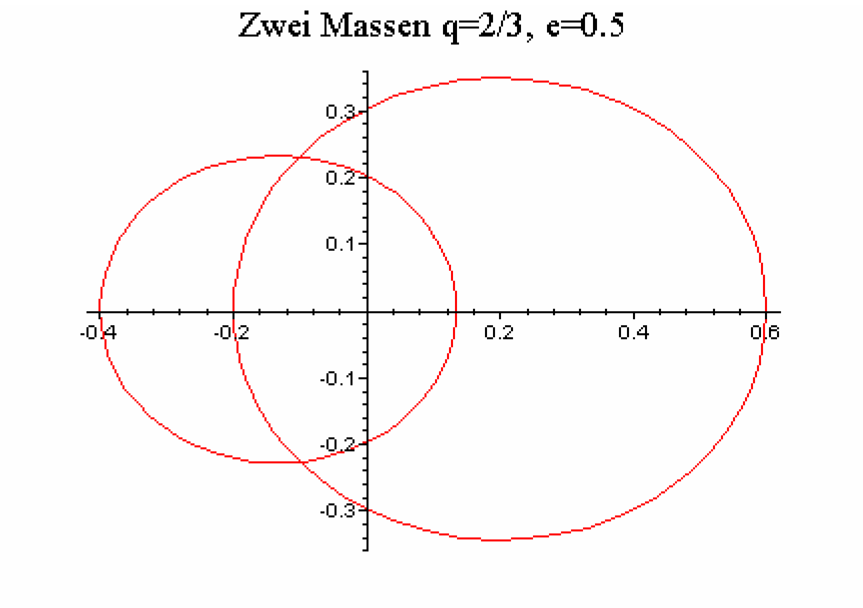

Fig. 3.1 shows the orbits of 2 masses with the ratio q = M_1/M_2 and the excentricity e = 0,5.

With the angular frequency $\omega = \dfrac{2\pi}{T}$ and the great semiaxes a_1 and a_2 the amplitudes of the radial velocitiy become

$$\upsilon_1 = a_1\omega \quad \text{and} \quad \upsilon_2 = a_2\omega \tag{3.2}$$

In many cases tidal friction has made the orbits nearly circular and diminished the excentricity “*e”* to practically zero. The ratio of the greater semiaxes a_1 and a_2 equals the mass ratio

$$\frac{a_1}{a_2} = \frac{M_2}{M_1} = \frac{\upsilon_{1r}}{\upsilon_{2r}} \tag{3.3}$$

The distance of the two components is

$$a = a_1 + a_2 = \frac{\upsilon_1 + \upsilon_2}{\omega} \tag{3.4}$$

With Kepler's 3rd law

$$M_1 + M_2 = \frac{a^3}{G}\omega^2 \tag{3.5}$$

this gives

$$M_1 + M_2 = \frac{\left(\upsilon_{1r} + \upsilon_{2r}\right)^3}{G\omega \cdot sin^3\, i} \tag{3.6}$$

Using (3.3) to eliminate υ_{2r} we can rewrite (3.6) in a very useful form

$$\frac{M_2^{\ 3}}{\left(M_1 + M_2\right)^2}\sin^3 i = \frac{\upsilon_1^{\ 3}}{2\pi G}T \tag{3.7}$$

Astrophysicists call the left side of (3.7) the ***mass function***. Usually the inclination is unknown. In those cases one takes the mean value over the halfsphere $\langle \sin^3 i \rangle = 0{,}60$. Very often only υ_{1r} and ω are known ("spectroscopic binaries"). If the two components are eclipsing each other the inclination may be determined from the light curve. The mass of the visible component is estimated from the optical spectra since the luminosity (the total radiation power) depends on the mass as follows

$$\frac{L}{L_{sol}} = \left[\frac{M}{M_{sol}}\right]^{3{,}5} \tag{3.8}$$

As an example I give the data of the x-ray binary **Cyg X 1**:
Distance from earth is about 6000 Ly; [1 Ly = $9{,}46\cdot 10^{12}$ km]
$(a_1 + a_2) = 0{,}2\ AU$ [1 astronomic unit (*AU*) = $149{,}598\cdot 10^{6}$ km]
Period *T* = 5,6 days

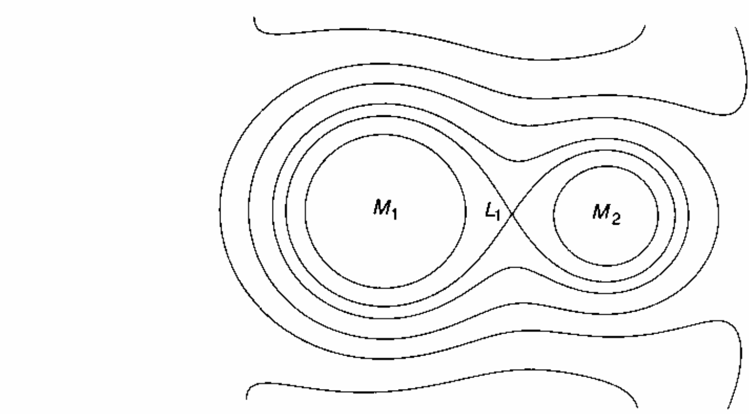

Fig. 3.2 a) A section through the equipotential surfaces of a binary star system with the components $M_1 = 2\ M_2$. The surfaces just touching at the L_1-point are called Roche lobes. The potential is $\varphi(r) = -\frac{GM_1}{|\vec{r} - \vec{r}_1|} - \frac{GM_2}{|\vec{r} - \vec{r}_2|} - \frac{1}{2}(\omega \times \vec{r})^2$**.**

The optical companion has $M_I \approx 40$ solar masses, spectral type OB 8
The BH $M_X \geq 10$ solar masses were estimated, its angular momentum J seems to be very small, the inclination is i ≈ 50°, $\sin^3 i \approx 0{,}76$.

The source of X-rays is the corona, the hot inner part of the accretion disk. Its matter is supplied from the visible star which is massive and produces a strong „plasma wind". It fills the "Roche lobe" (the lowest common equipotential surface near M_I) and is transferred through the point L_1 (s. Fig. 3.2.a) like a nozzle to the accretion disk of the BH (s. fig. 3.2.b).

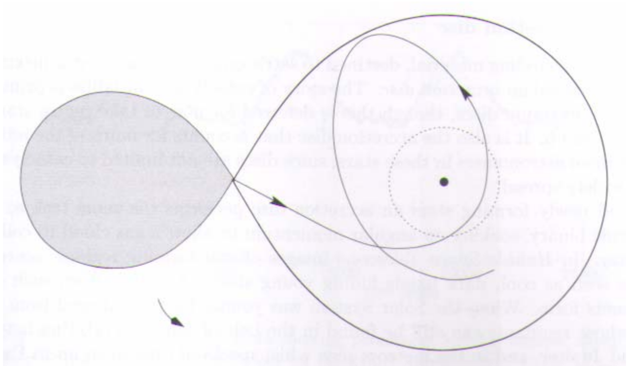

Fig.3.2 b) From the Roche lobe of the visible star at the left side streams matter into the accretion disk of the compact object at the right. A trajectory of matter emanating from the L_1-point is shown. The whole system is anticlockwise rotating. Credit: C. Helleir, Cataclysmic Variable Stars. Springer 2001.

3.2. Conservation laws in Schwarzschild metric. Gravitational redshift.

In order to study the orbit of a test particle (m << M) we go back to section 2.3 and consider equ. (2.4). We have used the Lagrange function $L = g_{ik}(x)\dot{x}^i\dot{x}^k$ and with its help found the geodesic equation. For the Schwarzschild metric we have

$$\frac{2L}{m^2} = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 \qquad (3.9)$$

Note that the m^2 in the denominator corresponds to a relativistic Lagrangian. Summing up its terms yields m^2c^2. Here the dot means differentiation $\dot{t} = dt/d\tau$ and $\dot{r} = dr/d\tau$ and

$$d\tau = \left(1 - \frac{r_S}{r}\right)^{1/2} dt \qquad (3.10)$$

where $d\tau$ is the lapse of proper time for a static observer in the gravitational field of the BH. Instead of τ any other affine parameter λ may be used. When observers far away from the BH ($r \to \infty$) check their clocks they would find them on equal speed, i.e. $dt = d\tau$. However, if one of the clocks is near the BH an observer far away finds its time units enlarged, that is $dt >> d\tau$.

$$dt = \frac{d\tau}{\left(1 - \frac{r_S}{r}\right)^{1/2}}, \qquad (3.11)$$

We may use optical frequencies of atomic emission or absorption as appropriate clocks and write ν_{em} for the frequency emitted locally at $r = r_{em}$ and ν_{rec} for frequency recorded at $r \to \infty$

$$\frac{\nu_{em}}{\nu_{rec}} = \frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{r_S}{r_{em}}\right)^{1/2}} \tag{3.12}$$

or for the red shifted frequency ν_{lab} received far away from the BH

$$\frac{\nu_{rec}}{\nu_{em}} = \left(1 - \frac{r_S}{r_{em}}\right)^{1/2} = \frac{\lambda_{em}}{\lambda_{rec}} = \frac{1}{z_g + 1} \tag{3.13}$$

where frequencies or wavelengths denoted by "lab" are laboratory values. z_g is called the gravitational redshift and should not be confused with the cosmological redshift z which is measured in the same way but is caused by cosmic expansion. At the event horizon $r = r_S$ the recorded frequency vanishes and the red shift becomes infinite. When the expected effects are small we may expand the square root and confirm the expression (1.5), formerly obtained from Newton's gravity in the first lecture, as an approximation.

We now write the Euler diff. equ. (2.5) for t, r, θ and φ respectively

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i} \quad \text{with } x^i = (t, r, \theta, \phi) \tag{3.14}$$

$$\frac{d}{d\lambda}(\dot{r}^2\dot{\theta}) = r^2 \sin\theta \cos\theta \dot{\phi}^2 \tag{3.15}$$

$$\frac{d}{d\lambda}(r^2 \sin^2\theta\dot{\phi}) = 0 \tag{3.16}$$

$$\frac{d}{d\lambda}\left[\left(1 - \frac{r_S}{r}\right)\dot{t}\right] = 0 \tag{3.17}$$

We assume $\theta = \pi/2$ and $\dot{\theta} = 0$ which restricts the motion to the equatorial plane. Instead of the r-equation we may use the relativistic energy-momentum relation

$$g_{kl}p^k p^l = m^2 \quad \text{or the velocities} \quad g_{kl}\dot{x}^k \dot{x}^l = 1 \tag{3.18}$$

which determines

$$L = \frac{m^2}{2} \tag{3.19}$$

It follows that Equ. (3.16) is zero and yields the conservation of angular momentum per unit mass

$$p_\phi = r^2\dot{\varphi} = const. = \frac{J}{m} = j \tag{3.20}$$

With (3.17) we obtain the conservation of energy

$$p_t = \left(1 - \frac{r_S}{r}\right)\dot{t} = const. = \frac{E}{m} = \varepsilon \tag{3.21}$$

Here ε is the energy per unit mass.

3.3. The model of gravitational collapse of Oppenheimer and Snyder.

We start with the Schwarzschild metric (2.23) and orient the coordinate system again so that $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. Then

$$ds^2 = d\tau^2 = \left(1 - \frac{r_S}{r}\right)dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2\dot{\phi}^2 \tag{3.22}$$

Division through $d\tau^2$ yields

$$\sigma = \left[\left(1 - \frac{r_S}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{r_S}{r}\right)^{-1} \cdot \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2\right] \tag{3.23}$$

We now introduce j and ε from (3.20) and (3.21)

$$\sigma = \left[\left(1 - \frac{r_S}{r}\right)^{-1} \cdot \varepsilon^2 - \left(1 - \frac{r_S}{r}\right)^{-1} \cdot \left(\frac{dr}{d\tau}\right)^2 - \frac{j^2}{r^2}\right] \tag{3.24}$$

with $\sigma = 1, 0$ for timelike and null geodesic respectively. We solve for $\dot{r}^2$ and choose $\sigma = 1$(matter with rest mass $m \neq 0$)

$$\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 - \left(\frac{j^2}{r^2} + 1\right)\left(1 - \frac{r_S}{r}\right) \tag{3.25}$$

With (3.24) we are now able to follow the work of Oppenheimer and Snyder (1939) who considered a very simplified case of gravitational collapse: a spherical star with mass M composed of matter with zero pressure (i.e. some kind of dust). The radial coordinate of the surface is $r = R(t)$. If the collapse procedes spherically we have $\phi = const.$ and $j = 0$ and the simplest geodesic would be

$$\left(\frac{dR}{d\tau}\right)^2 = \varepsilon^2 - 1 + \frac{r_S}{R} \tag{3.26}$$

where ε is in unites of mc^2 and velocity in unites of c. This equation describes the collapse witnessed by an observer at $r = R$. We assume $\varepsilon^2 << 0$ (bound state) and start with the local velocity

$$\left(\frac{dR}{d\tau}\right)^2 = 0 \quad \text{at} \quad R = R_{\max} = \frac{r_S}{1 - \varepsilon^2} \tag{3.27}$$

We find that the collapse velocity increases when r decreases. Surprisingly it passes smoothly through the event horizon at $r = r_S = 2M$, see fig. 3.3.

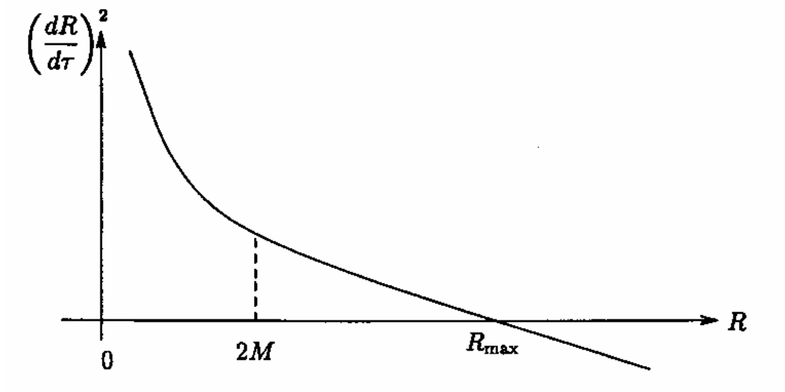

Fig. 3.3. The speed of spherical collapse witnessed by a local observer at the surface of a star (in units of c^2).Taken from K.P. Townsend gr-qc 970712.

The collapse takes the final time (without proof)

$$\tau = \frac{\pi r_s}{2c(1-\varepsilon)^{\frac{3}{2}}} \tag{3.28}$$

What does an observer see at $r \to \infty$? He or she observes dR/dt . From (3.24) we have after multiplication with $d\tau/dt$

$$\dot{R}^2 = \frac{1}{\varepsilon^2}\left(1-\frac{r_S}{R}\right)^2\left(\frac{r_S}{R}-1+\varepsilon^2\right) \tag{3.29}$$

This 3rd order function is plotted in fig.3.4.

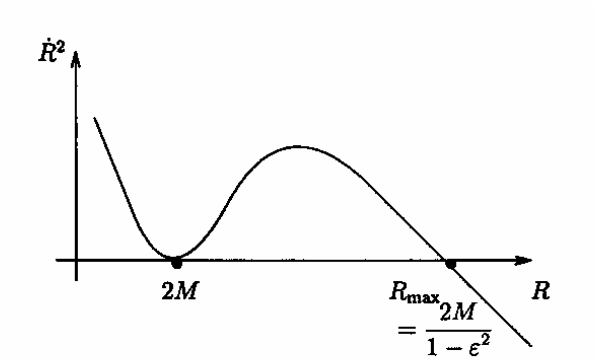

Fig. 3.4. Speed of collapse observed far away from the collapsing star. Taken from K.P. Townsend, gr-qc: 970712.

Surprisingly the speed does not increase monotonously but passes a maximum before it goes to zero at the event horizon. It is this phenomenon which led Oppenheimer and Snyder to the conclusion: “An external observer sees the star asymptotically shrinking to its gravitational radius (the radius r_S) “. Later this result gave the BH the name “frozen star”.. However, it can be shown that r_S also increases when a BH, which already exists, accretes a spherical shell of non vanishing mass and r_S covers the shell's radius in finite time (see S.-N. Zhang arXiv: 1003.1359 [gr-qc]).

3.4. Effective Potentials.

In Newton's gravitation the motion of a mass *m* orbiting around a large mass *M* (*m* << *M*) may be written in the form

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2+\frac{j^2}{2r^2}-\frac{GM}{r}=\varepsilon \tag{3.30}$$

This is the Kepler problem which we have been acquainted with in the Mechanics Lecture. The second term is the centrifugal term and may be added to the gravitational potential. The sum forms an effective potential

$$V_{eff}(r)=-\frac{GM}{r}+\frac{j^2}{2r^2} \tag{3.29}$$

We already know that there are 3 cases: 1) $\varepsilon > 0$ unbound hyperbolic orbits, 2) $\varepsilon < 0$ bound , elliptical or circular orbits, 3) $\varepsilon = 0$ is a limiting case with a parabolic orbit.

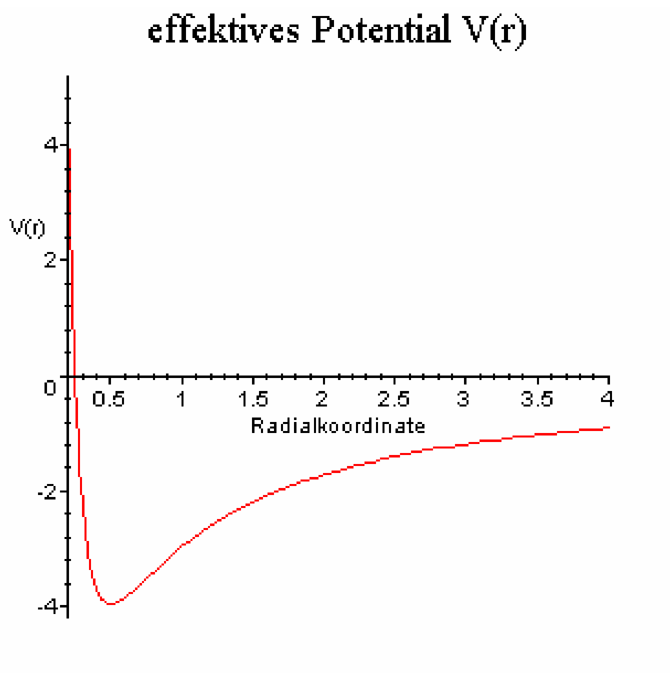

Fig. 3.5. A plot of the Newtonian effective potential of equ. (3.28)

A look at equ. (3.25) convinces us that an effective potential can even be derived from the Schwarzschild metric. For this purpose we rewrite (3.25) in the following form (for clarity *c* is explicitely written)

$$\frac{1}{2}\dot{r}^2-\frac{r_S}{2r}+\frac{j^2}{2c^2r^2}-\frac{j^2r_S}{2c^2r^3}=\frac{1}{2}\left(\varepsilon^2-1\right) \tag{3.30}$$

We introduce dimensionless quantities

$$\hat{r} = \frac{r}{r_S}, \quad a = \frac{j}{r_S\, c} = \frac{J}{r_S\, m\, c} \tag{3.31}$$

$$\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{\hat{r}} + \frac{a^2}{\hat{r}^2} - \frac{a^2}{\hat{r}^3} = \left(\varepsilon^2 - 1\right) \tag{3.32}$$

and obtain V_{eff}:

$$V_{eff}(\hat{r}) = 1 - \frac{1}{\hat{r}} + \frac{a^2}{\hat{r}^2} - \frac{a^2}{\hat{r}^3} \tag{3.33}$$

When $\hat{r}$ takes on large values ε approaches one, $\varepsilon \to 1$. We therefore may write approximately

$$\left(\varepsilon^2 - 1\right) \approx 2\left(\varepsilon - 1\right) \tag{3.34}$$

The right hand side of (3.30) becomes

$$\frac{1}{2}\left(\varepsilon^2 - 1\right) \approx \left(\varepsilon - 1\right) = \frac{E - mc^2}{mc^2} \tag{3.35}$$

This shows that for large values $\hat{r}$ the effective potential approaches the Newtonian value. The Newtonian referenz energy is $E = 0$ .This differs from Schwarzschild metric where we have approximately the reference $\varepsilon = E/mc^2 \approx 1$. The reason is that the relativistic energy also contains the rest mass energy.(s. fig. 3.5).

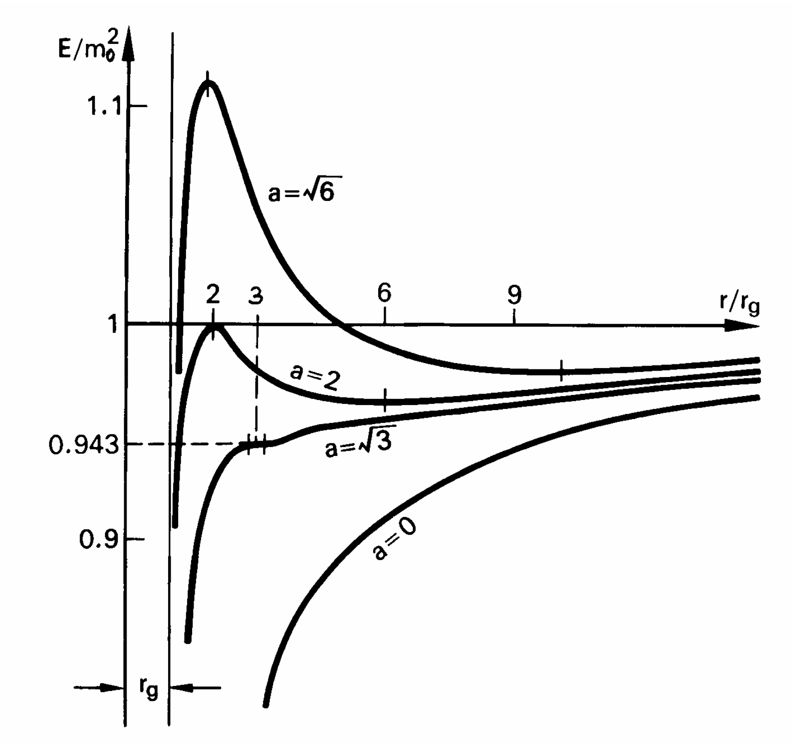

Fig. 3.5. The diagram shows of the relativistic effective potential (3.33) with the normalized angular momentum *a* as parameter. On the horizontal axis $r/r_g = r/r_S$ is plotted. The normalized angular momentum takes the values $a = 0,\ a = \sqrt{3},\ a = 2,\ a = \sqrt{6}$

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In order to reveal stable circular orbits we need to find the extrema of V_{eff}.

We differentiate (3.33) and find

$$\frac{\partial}{\partial r}V(\hat{r}) = \frac{1}{\hat{r}^2}\left[1 - \frac{2a^2}{\hat{r}} + \frac{3a^2}{\hat{r}^2}\right] = 0 \tag{3.36}$$

$$\hat{r}^2 - 2a^2 r + 3a^2 = 0 \quad \text{with the solutions} \quad \hat{r}_{1,2} = a^2\left(1 \pm \sqrt{1 - \frac{3}{a^2}}\right) \tag{3.37}$$

In what follows we discuss the solutions and their physical significance.

1) Motion without angular momentum $a = 0$ corresponds to radial collapse (3.36). $V'(\hat{r}) = 0$ when $\hat{r} \to \infty$ which we have already considered following Oppenheimer and Snyder..
2) A special limiting solution of (3.37) is when $a = \sqrt{3}$. Minimum and maximum coincide at $\hat{r}_{1,2} = 3$ or $r = 3r_S$ and provide the radius of a marginal stable orbit.
3) Stable orbits have $a \geq \sqrt{3}$. Two examples are shown in Fig. 3.5. One with $a = 2$ at $r_1 = 6r_S$, (the inner solution is $r_2 = 2r_S$), the other with $a = \sqrt{6}$ is $r_{1,2} = (6 \pm 3\sqrt{2}) \cdot r_S$ has a stable orbit at r_1.

We conclude that there are stable orbits for all $a^2 \geq 3$; $r > 3\,r_S$ but no stable orbits for all $r < 3\,r_S$. The last orbit is only marginally stable at $r = 3r_S$. We now calculate the corresponding energy for a marginal stable orbit ($\dot{r} = 0$) at $r = 3r_S$ using

$$-\frac{1}{\hat{r}} + \frac{a^2}{\hat{r}^2} - \frac{a^2}{\hat{r}^3} = \left(\varepsilon^2 - 1\right) \tag{3.38}$$

and obtain $\varepsilon^2 = \frac{8}{9}$. The binding energy in this innermost orbit is

$$E_B = \frac{mc^2 - \sqrt{\frac{8}{9}} \cdot mc^2}{mc^2} = 5{,}72\ \%\,\text{of } mc^2 \tag{3.39}$$

This is the maximal efficiency of mass-energy conversion near a static BH, approximately realized in Cyg X 1. Compare this with the energy gain from nuclear fusion. The maximal known gain (from pp-reaction) is comparatively small and only 0,7 %.of mc^2,

3.5. Orbits of zero mass particles.

Photons and neutrinos are considered as massless particles. We have to find the zero geodesics $ds^2 = 0$ and use for this case the equ. (322) and (3.23) where we set at the left side zero. We find

$$\varepsilon^2 - \dot{r}^2 - \frac{j^2}{r^2}\left(1 - \frac{r_S}{r}\right) = 0 \tag{3.40}$$

The specific energy ε and the angular momentum or spin j are conserved on geodesics. Therefore we treat both as constants. We divide (3.40) through j and introduce for the ratio

$$b = \frac{j}{\varepsilon}$$

the so called impact parameter. Instead of $d\tau$ we may go over to $j \cdot d\tau$. This is only a redefinition since j is constant on the orbits. We obtain from (3.40)

$$\dot{r}^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{r_S}{r}\right) \tag{3.41}$$

The effective potential is now

$$V_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{r_S}{r}\right) \tag{3.42}$$

From $\frac{\partial V_{eff}}{\partial r} = 0$ we find a maximum at $r = \frac{3}{2} r_S$ for the last (unstable) circular orbit (V has a maximum!). The energy of this orbit

$$\frac{1}{b^2} = \frac{1}{r^2}\left(1 - \frac{r_S}{r}\right) \quad \text{and} \quad b^2 = \frac{27}{4} r_S^2 \tag{3.43}$$

We will come back to photonic orbits when we come to ***gravitational lensing*** in lecture 12.

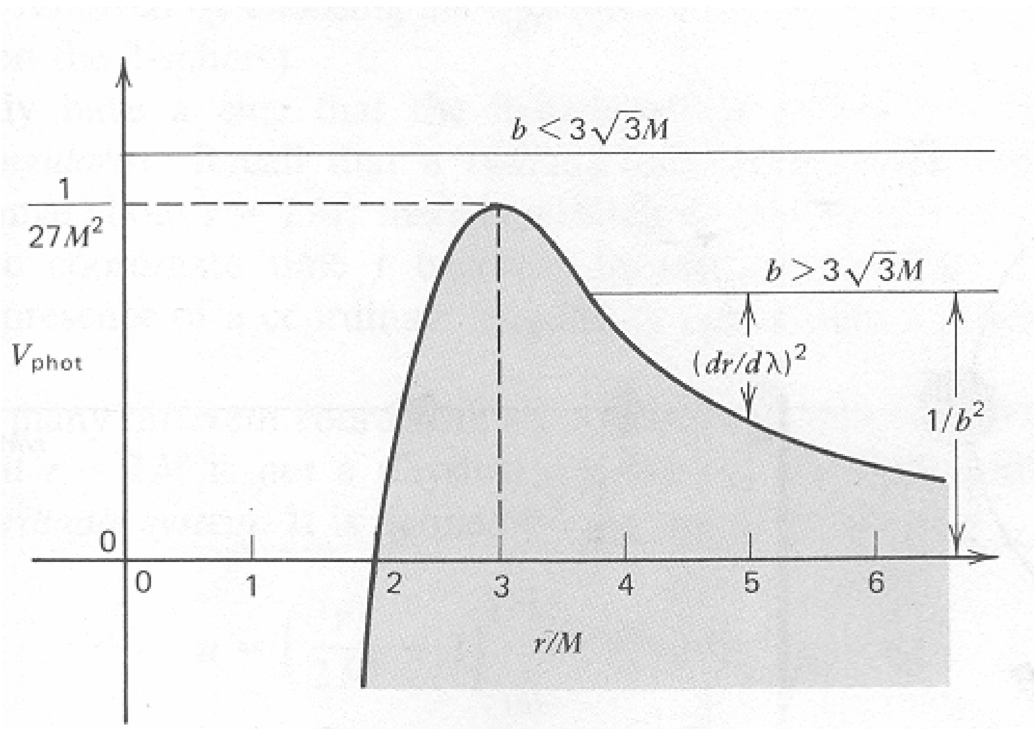

Fig. 3.6. Sketch of the potential of a particle with zero rest mass orbiting a static BH. $M = \frac{1}{2} r_S$ **.**

From Shapiro and Teukosy "Black Holes, White Dwarfs and Neutron Stars".1983 John Wiley & Sons.

3.5. Problems

3.5.1. Derive the effective potential of a zero mass particle (3.42), plot it and discuss its shape, use fig. 3.6 as a guide line.

3.5.2. Find the radius of a (stable or unstable) orbit from the extrema of $V_{eff}(r)$. Calculate the respective energy (3.43).

3.5.2. It is strange to think of photons circulating several times around the BH. Some photons may be backscattered and could in principle be observable. What would be the redshift factor in this case the maximum which a terrestrial observer could expect?