4. **Kruskal Coordinates and Penrose Diagrams.**

4.1. Removing a coordinate Singularity at the Schwarzschild Radius.

The Schwarzschild metric has a singularity at $r = r_S$ where $g_{00} \to 0$ and $g_{11} \to \infty$. However, we have already seen that a free falling observer acknowledges a smooth motion without any peculiarity when he passes the horizon. This suggests that the behaviour at the Schwarzschild radius is only a coordinate singularity which can be removed by using another more appropriate coordinate system. This is in GR always possible provided the transformation is smooth and differentiable, a consequence of the diffeomorphism of the spacetime manifold. Instead of the 4-dimensional Schwarzschild metric we study a 2-dimensional *t,r*-version. The spherical symmetry of the Schwarzschild BH guaranties that we do not loose generality.

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 \qquad (4.1)$$

To describe outgoing and ingoing null geodesics we divide through $d\lambda^2$ and set $ds^2 = 0$.

$$\left(1 - \frac{r_S}{r}\right)\dot{t}^2 - \left(1 - \frac{r_S}{r}\right)^{-1} \dot{r}^2 = 0 \qquad (4.2)$$

or rewritten

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{r_S}{r}\right)^{-2} \qquad (4.3)$$

Note that the angle of the light cone in *t,r*-coordinate.decreases when *r* approaches r_S
After integration the outgoing and ingoing null geodesics of Schwarzschild satisfy

$$t = \pm\, r^* + const. \qquad (4.4)$$

r^* is called "tortoise coordinate" and defined by

$$r^* = r + r_S \ln\left(\frac{r}{r_S} - 1\right) \qquad (4.5)$$

so that

$$\frac{dr^*}{dr} = \left(1 - \frac{r_S}{r}\right)^{-1}. \qquad (4.6)$$

As *r* ranges from r_S to ∞, r^* goes from -∞ to +∞. We introduce the null coordinates u, υ which have the direction of null geodesics by

$$\upsilon = t + r^* \quad \text{and} \quad u = t - r^* \qquad (4.7)$$

From (4.7) we obtain

$$dt = \frac{1}{2}(d\upsilon + du) \qquad (4.8)$$

and from (4.6)

$$dr = \left(1 - \frac{r_S}{r}\right) dr^* = \frac{1}{2}\left(1 - \frac{r_S}{r}\right)(d\upsilon - du) \tag{4.9}$$

Inserting (4.8) and (4.9) in (4.1) we find

$$ds^2 = \left(1 - \frac{r_S}{r}\right) du d\upsilon \tag{4.10}$$

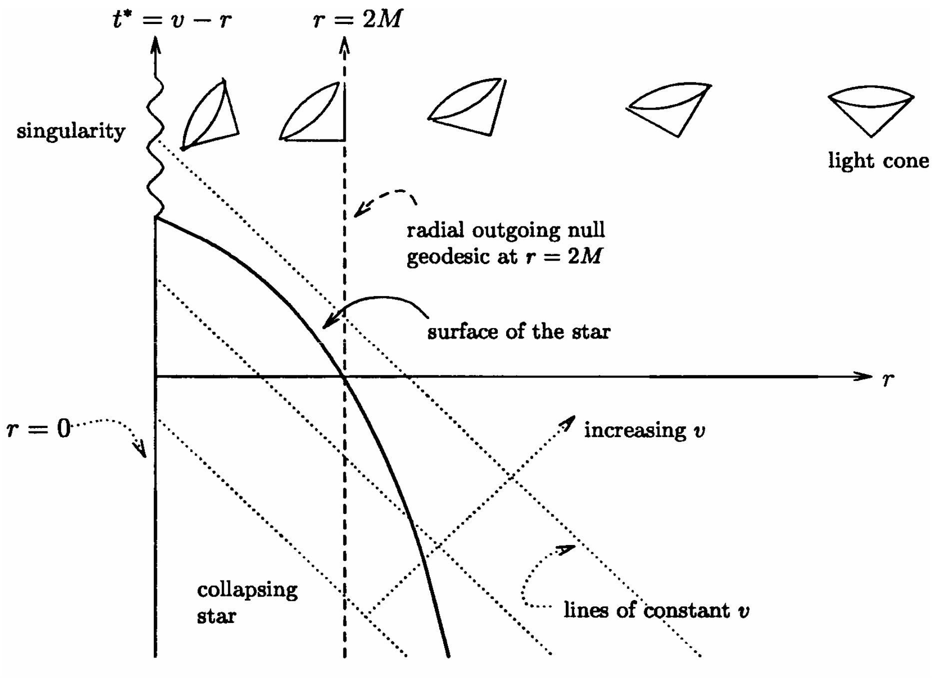

Fig. 4.1. This plot of $t = \upsilon - r$ versus r is called a *Finkelstein diagram.* When the surface of the star approaches $r \to r_S$ the light cones distort. Instead of outgoing null geodesics they coincide with the horizon. Therefore we can say: "the horizon is generated by the null geodesics".

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You will often find the ***Finkelstein diagram*** used to illustrate a collapsing star (David Finkelstein 1958). One may also add the φ-coordinate to construct a 3-dimensional diagram of the same kind.

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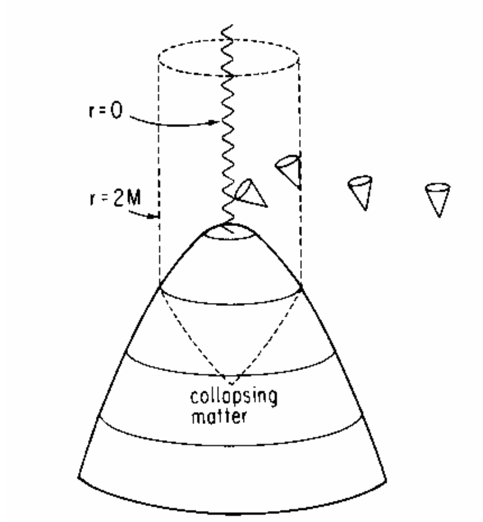

Fig. 4.2. Another representation of a collapsing star. Each circle in a section parallel to the r,φ-plane at t = constant is in reality a sphere.

4.2. Kruskal-Szegeres Coordinates.

Considering equ. (4.5) we find that r is now a function of u and υ

$$r^* = r + r_S \ln\left(\frac{r}{r_S} - 1\right) = \frac{1}{2}(\upsilon - u) \qquad (4.11)$$

We now rewrite the expression in parenthesis in (4.10) with

$$\left(1 - \frac{r_S}{r}\right) = \frac{r_S}{r}\left(\frac{r}{r_S} - 1\right) \qquad \rightarrow \qquad \frac{r}{r_S} + \ln\left(\frac{r}{r_S} - 1\right) = \frac{1}{2r_S}(\upsilon - u)$$

and use (4.11) to replace $\left(\frac{r}{r_S} - 1\right)$ by exponentials. We obtain

$$ds^2 = \frac{r_S \exp - {r}/{r_S}}{r} \cdot \exp(\upsilon - u)/2r_S \cdot du d\upsilon \qquad (4.12)$$

which is now of the form $ds^2 = g_{12}(r,u,\upsilon)\, du d\upsilon$. Note that g_{12} is non singular at $r = r_S$ while $u \rightarrow \infty$ and $\upsilon \rightarrow -\infty$. We may absorb the second exponential of (4.12) in the coordinates and define new coordinates

$$U = -\exp - {u}/{2r_S} \qquad \text{and} \qquad V = \exp {\upsilon}/{2r_S} \qquad (4.13)$$

The metric becomes now in terms of U and V

$$ds^2 = -\frac{4 r_S^{\,3}}{r} \exp - {r}/{r_S} \cdot dU\, dV \qquad (4.14)$$

The final transformation brings the ccordinates in the form

$$T = \frac{1}{2}(U + V) \quad \text{and} \quad X = \frac{1}{2}(V - U) \tag{4.15}$$

and the 2-dimensional metric becomes

$$ds^2 = \frac{2r_S^{\ 3}}{r}\exp - {r}/{r_S}\,(dT^2 - dX^2) \tag{4.16}$$

This metric was first introduced by Martin Kruskal nad George Szekeres in 1960. The relation between old (*t,r*) and new coordinates is as follows

$$\left(\frac{r}{r_S} - 1\right)\exp {r}/{r_S} = X^2 - T^2 \tag{4.17}$$

and

$$\frac{t}{r_S} = 2 \cdot tgh^{-1}\left({T}/{X}\right) \tag{4.18}$$

The Kruskal metric is initially defined for $T < 0$ and X > 0 but it can be extended by analytic continuation to $T > 0$ and $X < 0$. The former coordinate singularity $r = r_S$ corresponds in Kruskal coordinates to $UV = 0$, that is either $T = 0$ or $X = 0$. The singularity at $r = 0$ now corresponds to $TX = 1$ and is plotted as hyperbola with 2 branches in the 2[nd] and 4[th] region

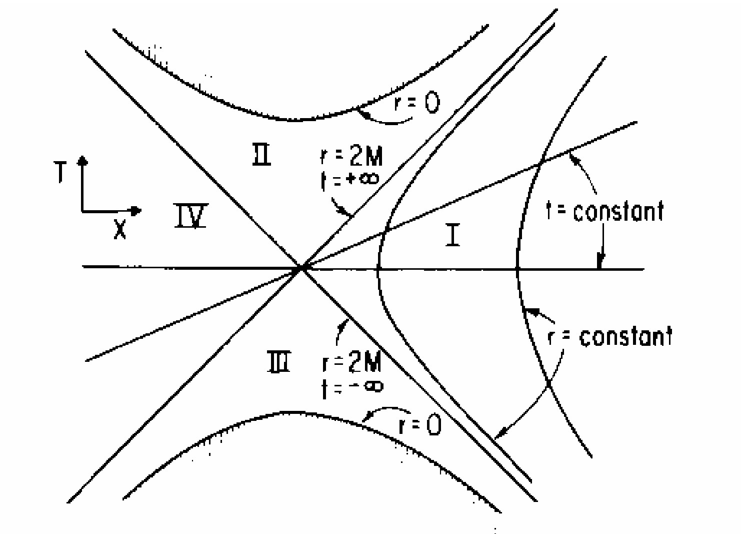

Fig. 4.3. The analytic extension of the Schwarzschild spacetime by Kruskal coordinates. Each point *r* = constant is a 2-sphere. It is represented in the diagram as hyperbolae with the X-axis as symmetry axis. Straight lines correspond to a constant time *t* . However, at the two 45° diagonal lines $r = r_S$ which represents a limiting case where a timelike line goes over in a spacelike line.

4.3. The surprising structure of the extended spacetime.

The structure of the ***extended Schwarzschild spacetime*** is divided in four regions:

1) Region I is the original spacetime which is observable by physical instruments. It is our world. Radial infalling matter crosses the hyperbolae and finally hits the line $T = X$ where it crosses the horizon.

2) Infalling matter enters region II (at $T = X$) and will fall into the singularity at $r = 0$. Any light signal from region II will remain there and also fall in the singularity. Region II describes the BH.

3) Region III is the time reversal of region II. An observer present in III must have been originated in the singularity and must leave region III again to region I. Therefore III is called a ***white hole.*** In the sixties some astronomers speculated that Quasars might be fuelled by white holes. However, observations at high resolution have unambiguously shown that the intense emission is due to matter which moves to the BH and finally vanishes there. Besides these observational evidences the existence of white holes would cause severe thermodynamic problems.

4) Region IV has properties identical with those of region I, i.e. represents an asymtotically flat region which lies inside (!!) of the radius $r = r_S$.

5) The singularity at $r = 0$ cannot be removed. The components of the Riemann curvature tensor diverge there as r^{-3} and tidal forces become infinitely large.

In the original Schwarzschild representation correct for $r > r_S$ the region IV spacetime is left out (see e.g. Fig. 4.5). We are going to illustrate this by embedding the relevant space into a 3-dimensional flat space. The metric in cylinder coordinates looks as follows

$$d\sigma^2 = dr^2 + dz^2 + r^2 d\phi^2 = dr^2\left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\phi^2 \qquad (4.19)$$

When we compare this with Schwarzschild

$$d\sigma^2 = dr^2\left(1 - \frac{r_S}{r}\right)^{-1} + r^2 d\phi^2$$

we find

$$\left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{r_S}{r}\right)^{-!} - 1 \qquad (4.20)$$

After integration the non-euclidian hypersurface (a 2d-hyperboloid) is embedded in the 3d euclidian space by

$$z = \pm\, 2r_S \cdot \sqrt{\left(\frac{r}{r_S} - 1\right)} \quad \text{for} \quad r > r_S \qquad (4.21)$$

All allowed points lie on the surface of the hyperboloid. The space points inside the horizon ($r < r_S$) are left out.

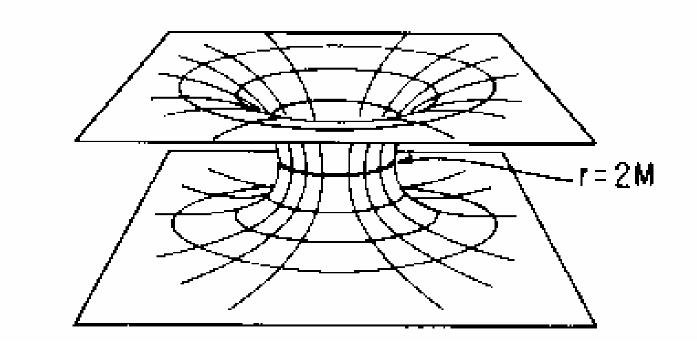

Fig.4.4. The spherical geometry of the hypersurface at t = 0 shown as it is embedded in flat space. The figure contains all space points $r \geq r_S$ but all points $r < r_S$ are lacking

Do we have to take this discussion seriously, when only region I is observationally accessible? We have to admit that all regions discussed so far are valid solutions of Einstein's equations for the Schwarzschild problem. But which of those solutions are realized in nature. In order to decide this question we certainly need observations but also the consideration of other physical laws which should not contradict those solutions. We will discuss related questions when we present the thermodynamics of BHs in lecture 7..

4.4. Penrose diagrams.

When you read papers on subjects concerned with GR or on a special metric you will often find that the causal structure is discussed in a Penrose diagram, which allows to consider the respective geometry in a compactified form. As an example we consider the Minkowski metric ($c = 1$)

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega = (dt + dr)\cdot(dt - dr) - r^2 d\Omega$$

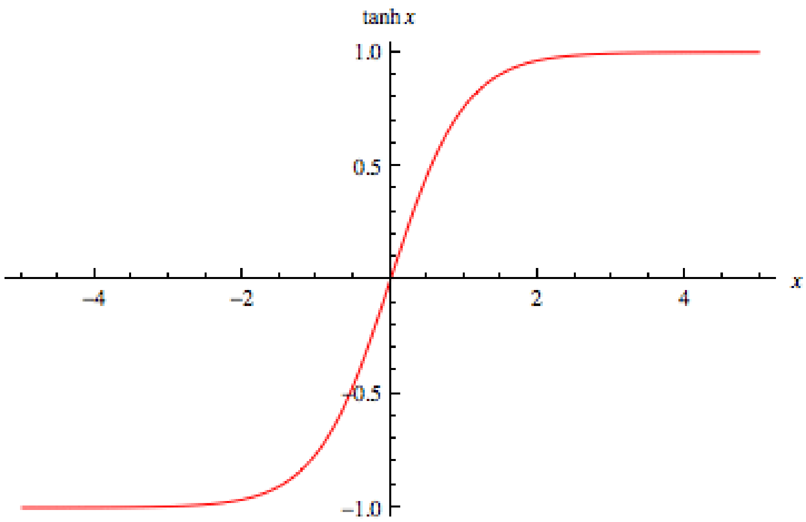

Fig. 4.5. A plot of the function tanh(x) which approaches +1 for $x \to \infty$ and -1 for $x \to -\infty$

The transformation to be found should

1) preserve the light cone and
2) map the entire infinite space to a finite portion of the 2d-plan

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The expressions $dt \pm dr = 0$ describe the propagation on the light cone. The transformation should have the form

$$Y^+ = F(t+r) \quad \text{and} \quad Y^- = F(t-r) \tag{4.22}$$

The function $\tanh(x)$ has the requested propertiy. Therefore we set

$$Y^+ = \tanh(t+r) \quad \text{and} \quad Y^- = \tanh(t-r) \tag{4.23}$$

The entire spacetime is mapped on the triangle bounded by

$Y^+ = 1$ with $(t+r) \to \infty$ in the diagram called I^+

and

$Y^- = -1$ with $(t-r) \to -\infty$ in the digram called I^-

We can transform the Kruskal diagram in the same way by applying the transformation of (4.23) to the coordinates T and X of equ. (4.16) and Fig. 4.3. Light ray (null geodesics) are going parallel to Y^+ and Y^-.

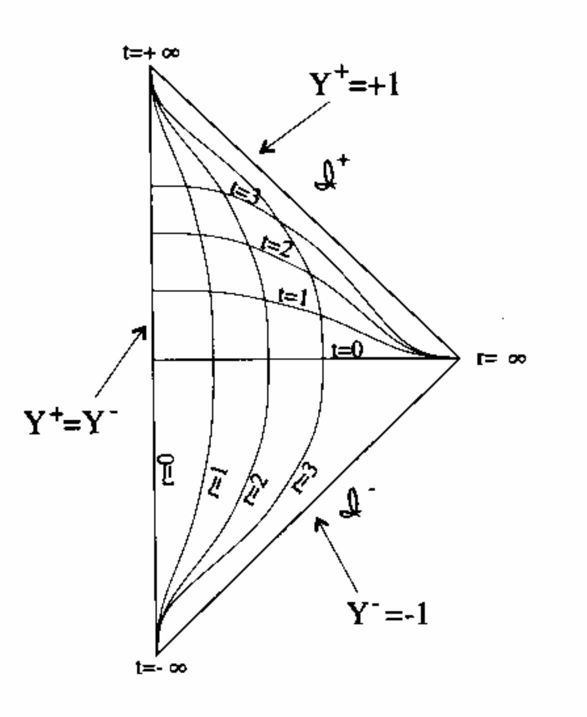

Fig. 4.6. The Carter-Penrose diagram of the Minkowski spacetime.

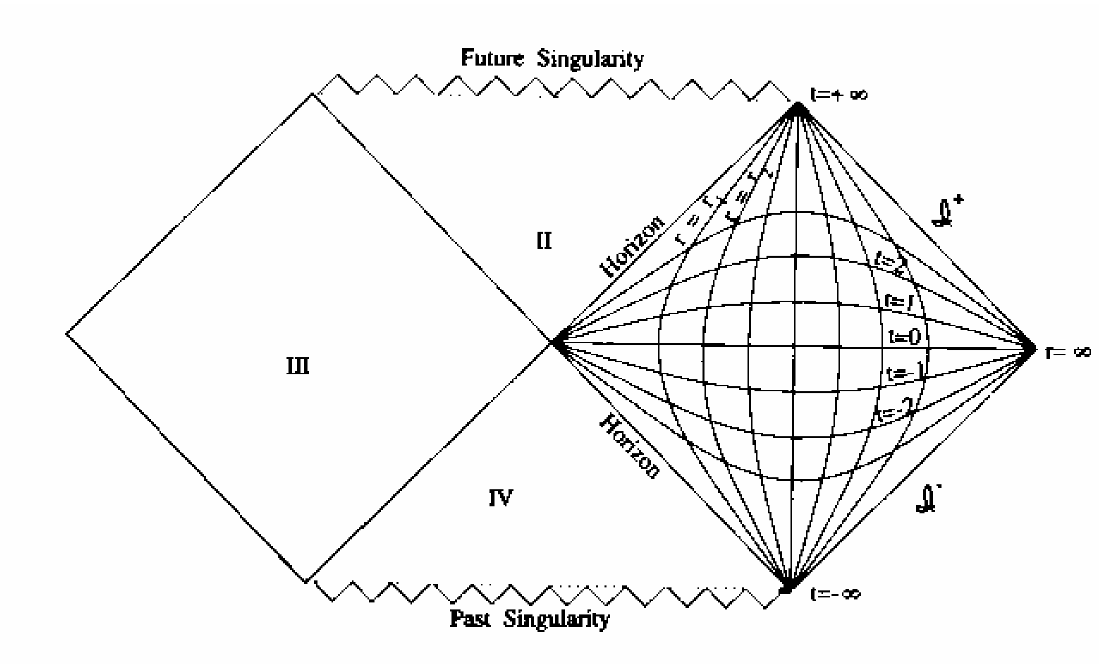

Fig. 4.7. The Carter-Penrose diagram of a static BH obtained from Kruskal coordinates as representated in Fig. 4.3. The axes assigned with t and r in the diagram correspond to *T* and *X* used in Section 4.2.

The infinities of the Kruskal diagram ($T \to \pm\infty$) appear in Fig.4.7. as finite points forming the upper and lower peaks. The hyperbolae of the singularity at $r = 0$ are compressed to a horizontal lines. The horizon is a global property and forms a lightlike surface which separates the spacetime in an inner and outer region. All events in the outer region (region I) can send signals (light rays) to I^+ and timelike trajectories to $T = \infty$. But any light ray which is emitted in the inner region (region II) will never reach the future asymptotic infinity ($T = \infty$) nor can matter reach the outer region I.

4.5. Problems

4.5.1. Consider the following metric

$$ds^2 = \frac{1}{t^4} dt^2 - dr^2 \qquad (4.24)$$

Is the singularity at $t = 0$ a coordinate singularity?
Hint: Use the transformation $\tilde{t} = t^{-1}$ to investigate the metric (4.24) and discuss the extension of coordinates. The spacetime geometry is geodesically complete, when all the geodesics approaching $t = 0$ extend to arbitrary large values of the affine parameter (e.g. τ).
4.5.2. Start with the radial collapse of fig.3.4. Make a hand drawing of the trajectory of a collapsing mass in the original Schwarzschild coordinates (*r,t*). Now make copies of the diagrams in fig.4.3 and fig. 4.7. Draw qualitatively the respective worldlines of infalling matter in Kruskal and in Penrose coordinates.
4.5.3. Find arguments why the existence if white holes appears implausible.