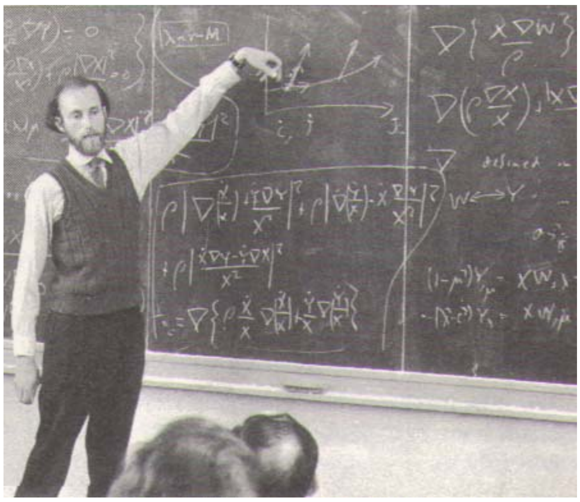
6. Kerr metric. Rotating Black Holes.

6.1. Historical remarks

The sixties and the seventies of the last century are considered today as the "Golden Age" of relativistic astrophysics and especially of black hole research. In the preceding decades general relativity (GR) had been a very special subject conducted mostly by mathematicians.
Since Schwarzschild's discovery (1916) and Friedmanns solutions (1922 and 1924) for an expanding universe nothing really interesting has happened which could have attracted the interest of physicists or astronomers. The research of Oppenheimer and his group in the late thirties had been interrupted by the war and was afterwards not taken up again. In the early sixties some new impetus stirred up. This development had many reasons. In Astronomy new instruments were available which could look much deeper into the universe than before and new mathematical tools in algebra and differential geometry had been developed. But probably most important the end of nuclear weapons research had released some first class physicists in Soviet Union (Zeldovich, Sacharow) and in the USA (Wheeler, Feynman). They were now back at universities looking out after attractive fields of fundamental research. New goups formed working on GR: Wheeler's group at Princeton, Penrose's and Sciama's at Cambridge, Zeldowich's at Moscow and Pascual Jordan's group (with Jürgen Ehlers and Engelbert Schückung) at Hamburg. Last not least: a group of mathematicians and theoretical physicists founded a new relativity group under the lead of Alfred Schild at the university of Texas in Austin. Roy Patrick Kerr (*1934), a young mathematician from New Zealand joined the group in 1962 (s. fig. 6.1). In 1963 Kerr published a two-pages Phys. Rev. Letter (**11**, p. 237. 1963) on a new solution of Einstein's equations which were valid outside of rotating masses. The full importance of this solution was not immediately clear.

Fig. 6.1.a) Roy Kerr (*1934). Picture taken about 1975.



b) Brendon Carter (*1942) lecturing at a summer school on BHs.

But soon after he red Kerr's letter Denis Sciama at Cambridge found a British research student, Brandon Carter (*1942), whom he asked to investigate Kerr's solution further. Within a year Carter was able to present a physical interpretation. The solution belongs to a rotating BH which is axial symmetric and stationary. If the BH has a non vanishing angular momentum then a section of space outside the horizon is twisted so that masses even when at rest are dragged around the BH's axis of rotation. Kerr gave his solution in carthesian

coordinates which is not appropriate for practical use. Today it is usually written in Boyer – Lindquist coordinates (1967). Robert Boyer joined also the Texas group in the sixties. Unfortunately he was killed with many others by a student of the US Marine Corps who ran amok and fired from the Austin university tower. And then a typical US-american scene: When this happened the students rushed to their dormitories and returned the fire.
A decade later, Chandrasekhar, the grand old man in astrophysics at Chicago, in his Ryerson Lecture of 1978, said: "**In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician Roy Kerr, provides the *absolutely exact representation* of untold numbers of massive black holes that populate the universe**".

6.2. The Kerr metric in Boyer-Lindquist coordinates.

In a realistic model of gravitational collapse the angular momentum of the progenitor star has to betaken into account. The angular momentum should be conserved and therefore a realistic BH should in general have an angular momentum. We are looking for a stationary solution written in t, r, θ, ϕ and choose the z-axis as axis of rotation. In order to limit the number of tensor components g_{kl} we apply time inversion $t \to -t$ which also changes the direction of rotation. That is the angular velocity $d\phi/dt$ also changes sign. When the transformations $t \to -t$ and $\phi \to -\phi$ applied together the metric is left invariant. This has the consequence that $g_{k0} = 0$ and $g_{n3} = 0$ for $k, n = 1, 2$. We may set $g_{12} = 0$ without loosing generality. What is left are the non-vanishing components $g_{00}, g_{03}, g_{11}, g_{22}, g_{33}$ which we all find again in the Kerr metric expressed in Boyer –Lindquist coordinates

$$\begin{aligned} ds^2 = &\left(1 - \frac{r_S r}{\Sigma}\right) dt^2 + \frac{2a \cdot r_S r \sin^2\theta}{\Sigma} dt d\varphi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 \\ &- \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta \cdot d\varphi^2 \end{aligned} \tag{6.1}$$

with

$$a \equiv \frac{J_{BH}}{Mc^2} \text{(BH angular momentum)}, \Delta \equiv r^2 - r_S r + a^2,\ \Sigma \equiv r^2 + a^2\cos^2\theta \tag{6.2}$$

The component of the metric tensor in mixed coordinates $g_{t\phi}$ has no time dependence $g_{t\phi}(r,t) = g_{t\phi}(r)$ as expected. Again we find a coordinate singularity in $g_{11} = g_{rr}$. When $\Delta = 0$ then $g_{rr} \to \infty$.
This occurs at the roots of the quadratic equation

$$\begin{gathered} \Delta \equiv r^2 - r_S r + a^2 = 0 \\ r_\pm^H = \frac{r_S}{2} \pm \frac{r_S}{2}\sqrt{1 - \frac{4a^2}{r_S^2}} \end{gathered} \tag{6.3}$$

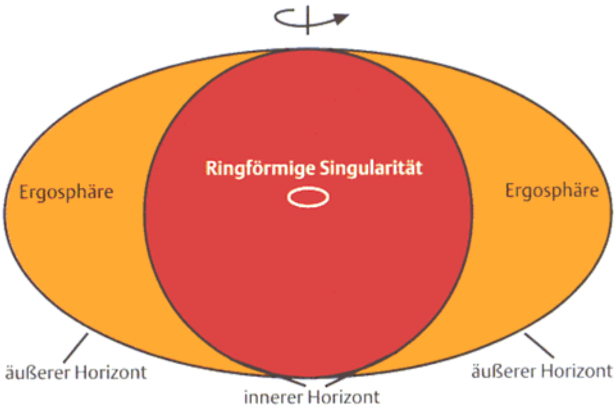


Fig. 6.2. The space nearest to a rotating BH is shown. The red sphere with radius r_+^H is the horizon. The orange section, outside limited by $r_+^{ES}(\theta)$ and inside by r_+^H is called ergosphere. It has its maximum extension when $\theta = \pi/2$. The solution r_-^H is a ring shaped singularity but is beyond the horizon and therefore without physical relevance.

(6.4)

Where $|a| \le \frac{r_S}{2}$ or $|a| \le M$ (here is G=c=1). The equal sign stands for a maximally rotating BH. In all cases with $|a| > M$ (6.3) has no real solution, the BH would have a gravitational field with no horizon, that means a "naked singularity". This unnatural solutions is forbidden due to the principle of ***cosmic censorship***, conjectured and first formulated by Roger Penrose (1965): "All physically reasonable space-times are globally hyperbolic, i.e. no singularity is ever visible to any observer". Unfortunately up to now the cosmic censorship has not been proved.

We now investigate the zeros of

$$g_{tt} = 1 - \frac{r_S r}{r^2 + a^2 \cdot \cos^2\theta} = 0 \tag{6.5}$$

$$r_\pm^{ES} = \frac{r_S}{2} \pm \frac{r_S}{2}\sqrt{1 - \frac{4a^2}{r_S^2}\cos^2\theta} \tag{6.6}$$

What is the physical meaning of the solutions r_+^H *and* r_+^{ES}? The first one is the radius of a sphere, the second one is an ellipsoid of rotation enclosing the sphere. At $\theta = 0, \pi$, where the spin axis intersects the sphere, both figures coincide (s. fig. 6.2). For $a > 0$ the solution r_-^H does not vanish, $r_-^H \ne 0$. It describes a ring-like singularity which is unobservable and therefore without physical relevance.

6.3. Geodesic motion. Energy and momentum conservation.

We will apply the same approach as in section 3.2. Again we set for a mass *m* moving along a geodesic

$$\frac{2L}{m^2} = \left(\frac{ds}{d\tau}\right)^2 = \left(1 - \frac{r_S r}{\Sigma}\right)\dot{t}^2 + \frac{2a \cdot r_S r \sin^2\theta}{\Sigma}\dot{t}\,\dot{\varphi} - \frac{\Sigma}{\Delta}\dot{r}^2 - \Sigma\dot{\theta}^2$$
$$-\left(r^2 + a^2 + \frac{a^2 r_S r \sin^2\theta}{\Sigma}\right)\sin^2\theta \cdot \dot{\varphi}^2 \tag{6.7}$$

and find energy and angular momentum

$$\frac{\partial L}{\partial \dot{x}^0} = \frac{\partial L}{\partial \dot{t}} = E \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^3} = \frac{\partial L}{\partial \dot{\phi}} = J \tag{6.8}$$

Then the energy density becomes

$$\frac{E}{m} = \varepsilon = \left(1 - \frac{r_S r}{\Sigma}\right)\dot{t} + \frac{a\, r_S\, r \sin^2\theta}{\Sigma}\dot{\phi} \tag{6.9}$$

and the specific angular momentum

$$\frac{J}{m} = j = -\frac{a\, r_S\, r \sin^2\theta}{\Sigma}\dot{t} + \left(\frac{\left[r^2 + a^2\right]^2 - \Delta\, a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta \cdot \dot{\varphi} \tag{6.10}$$

We have an additional relation

$$m^2 = g_{nm} p^n p^m \quad \text{or} \quad \kappa = g_{kl}\dot{x}^k \dot{x}^l$$

where κ=1 stands for time-like, κ=0 for null and κ= -1 for spacelike geodesics. We insert ε and *j* into (6.7) assuming as in section 3.2. that $\theta = \frac{\pi}{2},\ \dot{\theta} = 0$. We solve for $\frac{\dot{r}^2}{2}$ and find

$$\frac{\dot{r}^2}{2} + V_{eff}(r) = 0 \tag{6.11}$$

with

$$V_{eff}(r) = -\kappa\frac{r_s}{2r} + \frac{j^2}{2r^2} + \frac{1}{2}(\kappa - \varepsilon^2)\left(1 + \frac{a^2}{r^2}\right) - \frac{r_S}{2r^3}(j - a\varepsilon) \tag{6.12}$$

The mathematics is analogue to that applied in the case of Schjwarzsschild BHs. For physicists there is nothing new to be learned. Therefore we do not explicitly go through the complicated algebra.

6.4. The Ergosphere.

We will now show that a mass at rest inside the ergosphere but outside of the horizon $r_+^S < r < r_+^{ES}$ share the rotation of the BH. For this purpose we set $r, \phi = const.$ and write the coordinate angular velocity Ω in terms of the 4-velocity u^k. This gives

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} \qquad (6.13)$$

with the additional condition $g_{kl}u^k u^l = 1$ or

$$\left[g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}(u^\phi)^2\right] = 1 \qquad (6.14)$$

Division through $(u^t)^2$ yields

$$\left[g_{tt} + 2\Omega g_{t\phi} + g_{\phi\phi}\Omega^2\right] = \frac{1}{(u^t)^2} \qquad (6.15)$$

We expect Ω to be found between the two limits $\Omega_{\min} < \Omega < \Omega_{\max}$ which we determine as solutions of the following quadratic equation

$$\left[g_{tt} + 2\Omega g_{t\phi} + g_{\phi\phi}\Omega^2\right] \cong 0 \qquad (6.16)$$

with

$$\Omega_{\min,\max} = \frac{-g_{t\varphi} \pm (g_{t\varphi}{}^2 - g_{tt}g_{\varphi\varphi})^{1/2}}{g_{\varphi\varphi}} \qquad (6.17)$$

This is the case for the positive sign of the square root and $g_{tt} = 0$ which leads us back to equ. (6.5) and the outer surface i.e. $\Omega_{\min} = 0$ when $g_{tt} = 0 = r^2 - r_S r + a^2 \cos^2\theta$. Then we are leaving the rotating space and a test mass *m* may again be at rest. We conclude from these considerations that with $a \neq 0$ the space within the ergosphere (i.e. the coordinates) corotates with the BH. Fig. 6.3 shows an illustration of the twisted space in the ergosphere where a similar plot as in fig. 4.4. is used.

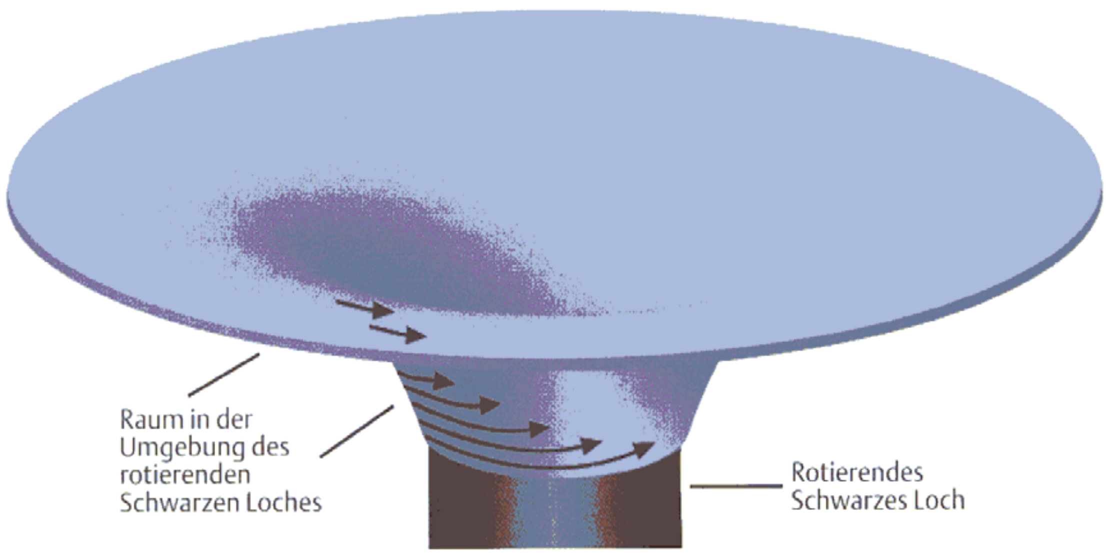


Fig. 6.3. This figure maps the corotation of space points of the equatorial 2-plane of the

ergosphere on an embedding 3-hyperspace.

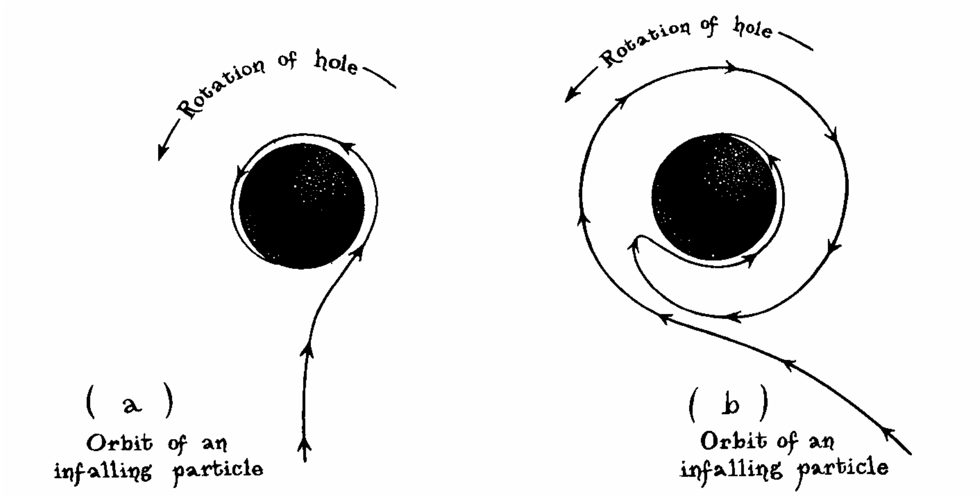


Fig. 6.4. Two trajectories of particle in the ergosphere are shown. In (a) the BH rotates in a left motion. The particle, although initially at rest must follow the BH rotation before it finally plunges in th BH. At the right side (b) the particle moves first in a retrograde orbit but the then it changes direction and must also follow the BH rotation before it disappears behind the horizon. From Kip S. Thorn, “Black holes and time warps”. New York, London 1994

6.5. Circular orbits

The conditions for motion in circular orbits should be applied to equ.(6.11). From $\dot{r} = 0$ we are left with the effective potential $V_{eff}(r)$ given in (6.12) and the requirements

$$V_{eff}(r) = 0 \quad \text{and} \quad \frac{\partial V_{eff}}{\partial r} = 0\,. \tag{6.18}$$

This leads for null geodesics i. e. for ***photon orbits*** ($\kappa = 0$) to a cubic equation in $\sqrt{r}$

$$r^2 - r_S \pm 2a\sqrt{\frac{r_S r}{2}} = 0 \tag{6.19}$$

with the solution

$$r_{ph} = r_S\left[1 + \cos\left\{\frac{2}{3}\arccos\left(\pm\frac{2a}{r_S}\right)\right\}\right] \tag{6.20}$$

Without rotation $a = 0$ this gives $r_{ph} = \frac{3}{2} r_S$, the Schwarzschild solution for photons.

For maximum angular momentum $a = M = \frac{r_S}{2}$ we have from equ. (6.20) $r_{ph} = \frac{1}{2} r_S$ $(corotation)$ and. $r_{ph} = 2 r_S$ $(counterrotation)$. The effective potential (6.12) yields with ($\kappa = 0$) and the conditions (6.18) a further relation

$$J = E\sqrt{3r_{ph}^2 + a^2} \qquad (6.21)$$

The photon orbit is also the innermost boundary of all circular orbits, that is of particle orbits on time-like geodesics ($\kappa = 1$). Unbound circular orbits $r > r_{ph}$ have energies $\varepsilon > 0$. Such particles will with a small outward perturbation escape to infinity. For all cases with $\varepsilon = 1$ which are related to parabolic orbits (and to $\varepsilon = 0$ in Newtonian mechanics) there is a **marginally bound** circular orbit at

$$r_{mb} = 2M \mp a + \sqrt{4M(M \mp a)} \quad \text{or} \quad r_{mb} = r_S \mp a + \sqrt{2r_S\left(\frac{r_S}{2} \mp a\right)} \qquad (6.22)$$

r_{mb} is also the minimum periastron distance of all parabolic orbits ($\varepsilon = 1$). A parabolic trajectory which penetrates the circle $r < r_{mb}$ must plunge into the BH.

We have just seen that not all circular orbits are stable. Instead the following requirements have to be fulfilled

$$V_{eff}(r) = 0 \ , \quad \frac{\partial V_{eff}}{\partial r} = 0 \quad \text{and} \quad \frac{\partial^2 V_{eff}}{\partial r^2} \geq 0 \qquad (6.22)$$

. One obtains from $V_{eff}(r)$ (s. equ. 6.12)

$$1 - \varepsilon^2 \geq \frac{r_S}{3r} \qquad (6.23)$$

Substituting the solution $\varepsilon = \frac{E}{m}$ for ($\kappa = 1$) in $V_{eff}(r)$ we obtain a quartic equation in $\sqrt{r}$.

The solution gives the ***innermost stable orbit*** also called the ***marginally stable orbit*** r_{ms}

$$r_{ms} = \frac{r_S}{2}\left\{3 + Z_2 \mp \left[(3 - Z_1)(3 + Z_1 + 2Z_2)\right]^{\frac{1}{2}}\right\} \qquad (6.24)$$

with

$$Z_1 \equiv 1 + \left(1 - \frac{4a^2}{r_S^2}\right)^{\frac{1}{3}}\left[\left(1 + \frac{2a}{r_S}\right)^{\frac{1}{3}} + \left(1 - \frac{2a}{r_S}\right)^{\frac{1}{3}}\right] \qquad (6.25)$$

and

$$Z_2 \equiv \left(\frac{12a^2}{r_S^2} + Z_1^2\right)^{\frac{1}{2}} \qquad (6.26)$$

For $a = 0$ we find $r_{ms} = 6M = 3r_S$ an for $a = \frac{r_S}{2} = \mathrm{M}$ we have $r_{ms} = \frac{r_S}{2}$ in the corotating case and $r_{ms} = \frac{9}{2} r_S$ for the counter-rotation; (s. also fig. 6.5. for other values of a).

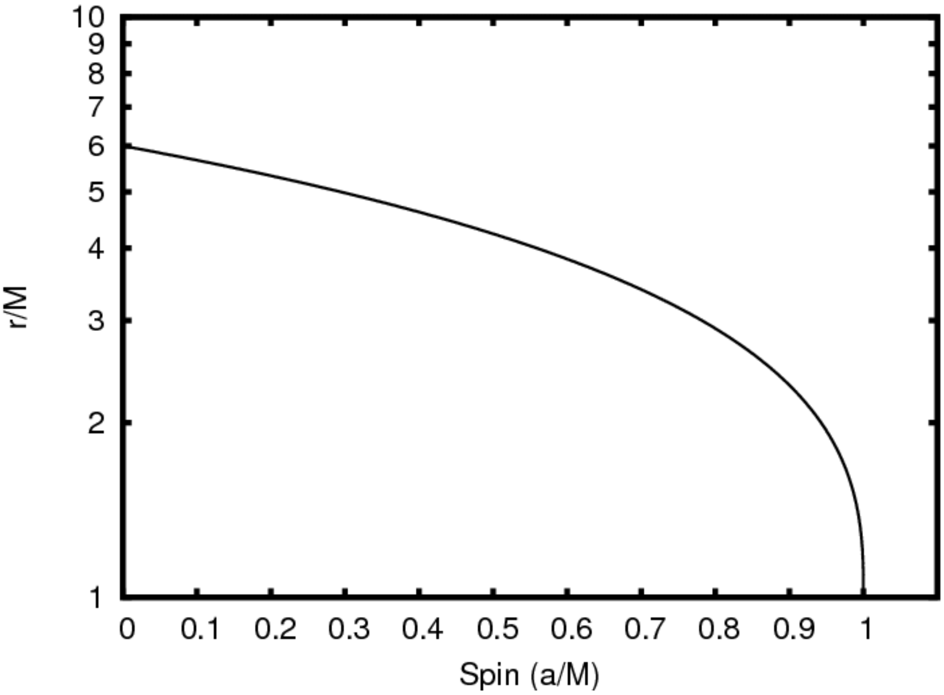


Fig. 6.5. Radius of innermost stable orbit r_{ms} (in units of $r_S / 2 = M$) for corotating particles as a function of spin (a in units of M). From Bardeen et al. 1972.

When in the accretion process gravitational energy is converted into radiation we like to know the efficiency η which is, as we have seen, a fraction of the rest mass

$$E_{rad} = \eta \cdot mc^2 \tag{6.27}$$

In order to find η we need the binding energy ε_{ms} of the inner most stable orbit

$$\eta \equiv 1 - \varepsilon_{ms} \tag{6.28}$$

We find ε_{ms} using (6.23) now with the equality sign

$$\eta = 1 - \left(1 - \frac{r_S}{3r_{ms}}\right)^{1/2} \tag{6.29}$$

ε_{ms} decreases for corotating motion from $\sqrt{\frac{8}{9}}$ $(a=0)$ to $\sqrt{\frac{1}{3}}$ $(a=M)$, while it increases in the counter-rotating cases from $\sqrt{\frac{8}{9}}$ $(a=0)$ to $\sqrt{\frac{25}{27}}$ $(a=M)$. The maximum binding The maximum efficiency is found in the corotating orbit with

$$\eta = 1 - 1/\sqrt{3} = 42{,}3\% \text{ of } mc^2 \tag{6.30}$$

This is the maximum possible conversion of mass into energy near a BH. Realistically it may be a fraction of about 30%. But even this is a huge efficiency of radiative energy production compared with fusion energy (η ≈ 0,7 %).

Now you have learned of so many different cases that you may feel a bit confused. With the following three tables I try to help you keeping things apart. There is one table on the radius of the horizon and the ergosphere, the other two on massive and massless particle orbits.

Schwarzschild BH	Angular momentum $a=0$	Horizon $r^H = r_S = 2M$	Ergosphere and static limit
Kerr BH $a \neq 0$	$a = M$	$r^H = \frac{1}{2} r_S = M$	$\frac{1}{2} r_S \leq r^{ES} \leq r_S$

Table 6.1.

Particles with rest mass $m \neq 0$

Angular momentum of BH	$a=0$	$a=M$, corotating	$a=M$, retrograde
Innermost stable orbit of particles	$3r_s$	$\frac{1}{2} r_s$	$\frac{9}{2} r_s$
Energy in units of mc^2	$\sqrt{8/9}$	$\sqrt{1/3}$	$\sqrt{25/27}$
Bindungenergie ε_{ms}	0,057	0,423	0,037

Table 6.2

Massless particles $m = 0$ (photons and neutrinos)

Angular momentum of BH	$a=0$	$a = \frac{1}{2} r_s$, corotating	$a = \frac{1}{2} r_s$, retrograde
Innermost marginally stable orbit	$\frac{3}{2} r_s$	$\frac{1}{2} r_s$	$2r_s$

Table 6.3.

6.5. The idee of Roger Penrose.

Is it possible to attract energy from a rotating BH? This question was first considered by Roger Penrose (he is also known as thesis advicer of Stephen W. Hawking). Within the ergosphere there exist particle trajectories with negative energy. Penrose used this property and started a thought experiment: A trajectory is chosen to penetrate the static limit. A particle with energy E_{in} is injected in the ergosphere. Then the particle splits in two. One piece moves with energy E_- along a negative energy trajectory (which is proofed to exist) down into the BH. The other comes out and is expelled with energy E_+. Energy conservation now gives

$$E_{in} = E_- + E_+ \quad \text{or} \quad E_+ = E_{in} - E_- \tag{6.31}$$

and since

$$E_- < 0 \text{ we have } E_+ > E_{in} \tag{6.32}$$

Penrose's proposal although logically correct seems to be unlikely to happen under realistic astrophysical conditions. Still there may be other processes possible where magnetic fields are involved. This has already been mentioned in connection with the appearance of jets. For more detailed information we must refer to the literature (s. e.g. Diss. Andreas Müller).

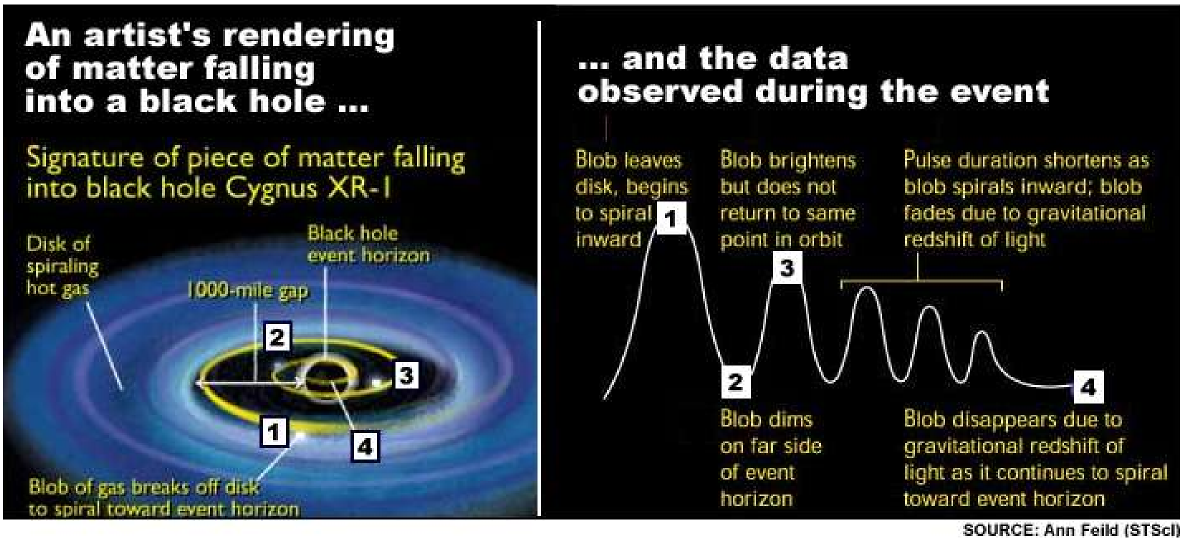


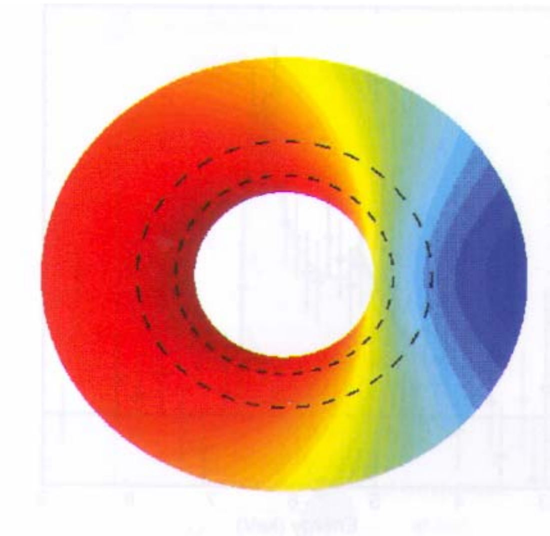
Fig. 6.6. Observation of an orbiting matter blob. Its periodic optical (or X-ray) signal fades due to gravitational redshift.

6.6. Is there a chance to estimate the BH's spin?

We have lerned about the Kerr metric and we refered already to Chandrasekhar who considered the Kerr BHs the most general model of all realistic BHs. They are indeed very simple objects which are characterized by only two parameters, the mass M and the spin a. The mass we have seen can determined either by motions in a binary system or by the

Eddington luminosity L_E. While this seems to be in many systems difficult enough the determination of the spin is even more complicated. There are several possible ways.

1) The first one relies on the determination of the radius (fig. 6.5) of the innermost orbit (e.g. observation of a decaying periodic signals s. fig. 6.6). The test mass would be a blob of accreted matter (at a SMBH even a star) which emits a strong enough signal periodically before it plunges into the hole. Such an observation with a 17 minutes period was reported from Sgr A* by R. Genzel et al. (2003 Nature , **425**, 934) In galactic binary systems such signals are not always correctly interpreted. One would first try to associate periodic signals with neutron stars. If this can be excluded the next check should exclude a flare (a hot eruption in the plasma of the inner section of the accretion disk.
2) A second indirect method uses the broad Kα emission of iron. The laboratory energy of this emission peaks at $h\nu = 6{,}35\ keV$ .The fast rotation of the innermost plasma ring broadens the line by the Doppler effect. Since the intensity of the red shifted part is enhanced over the blue shifted side this gives it an asymmetric shape. Furthermore the whole profile is shifted to lower energies by the gravitational redshift.



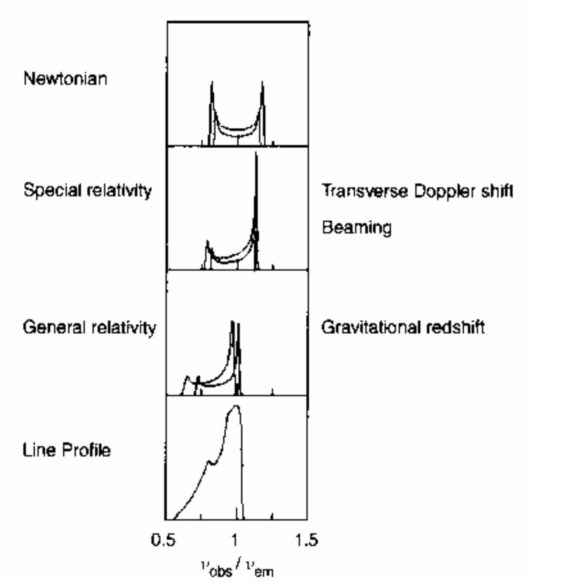


Fig. 6.5. Left: Illustration of the Doppler effect seen from the vertical direction in the plane of the paper. Credit: Peter Schneider, Extragalactic Astronomy and Cosmology. Berlin, Heidelberg 2006

Right: A schematic synthesis depicting all physical processes which contribute to the observed broadened and shifted profile of the Fe Kα – line.

3) The spin can also be measured by modelling the black body continuum spectrum of the accretion disk. In this method one determines essentially the innermost radius r_{bm} (s. section 6.5) by spectral temperature. One assumes that the inner edge of the accretion disk r_{in} is identical with r_{bm} an assumption which seems to be well supported in the high accretion state by many observations. Angular momentum values found in galactic binaries with a BH component and in microquasars range from $0{,}1 \le a \le 0{,}98\,M$. For this method the mass of the BH, the inclination angle and the distance from earth should be known with sufficient accuracy.

4) The radiation emitted from accretion disks shows (besides noise) sometimes so called quasi periodic oscillations (QPO) of some 100 Hz in galactic systems which are probably excited by a hydrodynamic or magneto-hydrodynamic mechanism. Usually a 2:3 - ratio is found for the lowest frequencies. There seems to be a nonlinear coupling to the rotating BH. Thus in special cases a conclusion about mass and spin may be possible. Shaposhnikov and Titarchuk used this method 2008 to determine the BH mass in the system XTE J1650-500 and could thus report the smallest BH mass known up to now.

6.7. Problems

6.7.1. Determine the radius $r_{mb}(a)$ of the innermost stable circular orbit around a Kerr BH for normal matter and for the angular momentum $a = 0, \pm a = M/2, \pm a = 3M/4, \pm a = M$.

6.7.2. Are a > M reasonable solutions?

6.7.3. The mass of a BH in a X-ray binary system is $10\ M_{sol}$. The luminosity of the disk was measured to be $0{,}1 \cdot L_E$. Calculate the accretion rate $\dot{M}$ (take $r = 2{,}2\, r_S$). In the high state of accretion the inner edge of the accretion disk was determined from the black body radiation to be $r_{in} = 2{,}2\, r_{S_i}$. What is the angular momentum a ? What is the maximum temperature of the disk?

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