7. BH thermodynamics. A step towards quantum gravity?

7.1. The "no – hair - theorem".

We may amend Chandrasekhar's remark (s. Ch. 6.1.) about the generality of the Kerr solution by saying that the Kerr BH is completely determined by 2 parameters, the mass M and the angular momentum $a = J/Mc$. We should add even a 3rd parameter, the electric charge. The metric of a charged static BH is called Reissner-Nordsröm metric (H. Reissner 1916, G. Nordström 1918), which we easily find from Schwarzschild by replacing $-\frac{r_S}{r}$ with $-\frac{r_S}{r}-\frac{Q^2}{r^2}$. In practice a charged BH plays no role in astrophysics. It might be difficult to imagine a situation that will lead to a charged BH. A plasma is usually neutral and even a local charge fluctuation may not last long enough to have any effect when a BH is formed. Nevertheless, we finally end up with three parameters: M, J and Q. It is a well proved statement that M, J and Q uniquely characterize a BH. Shapiro and Teukolsky go on: "*All other information about the initial state is radiated away in form of electromagnetic and gravitational waves during the collapse. The remaining three parameters are the only independent observable quantities that characterize a stationary BH. (B.Carter 1979).This situation is summerized by J.A. Wheeler's aphorism,* ***A black hole has no hair*** *".*

7.2. Hawking's Area Theorem.

Stephen Hawking proved the following theorem: ***The surface area of a BH***

$$A = 4\pi r^2\Big|_{horizon} \tag{7.1}$$

can never decrease whatever interaction it undergoes. If several BHs are present it is the sum of their surface areas that can never decrease. A Schwarzschild BH has

$$A = 4\pi r_S^2 = 16\pi M^2 \tag{7.2}$$

Now assume that two BHs M_1 and M_2 merges to one object M_3, with the mass $M_3 = M_1 + M_2$. The area becomes

$$A_3 = 4\pi r_{3S}^2 = 4\pi\cdot(2M_3)^2 = 4\pi\cdot 4(M_1+M_2)^2 \tag{7.3}$$

This is larger than the sum of the areas of the two distintly separated BHs

$$A_1 + A_2 = 4\pi\cdot 4(M_1^2+M_2^2) \tag{7.4}$$

That is

$$A_3 > A_1 + A_2 \tag{7.5}$$

The area gain is $\Delta A = 4\pi\cdot 8 M_1 M_2$. The relation between the increase of mass or energy and area for the Schwarzschild BH is from (7.3)

$$dM = \frac{1}{16\pi M} dA \qquad (7.6)$$

or more generally

$$dM = \frac{\kappa}{8\pi} dA \qquad (7.7)$$

where κ is called surface gravity. Physically it is the acceleration (exerted at infinity) to keep a mass at the horizon. κ is always constant over the horizon, a general statement which is sometimes called the ***zeroth law of BH thermodynamics***. We will later see that κ is proportional to the temperature of the BH.

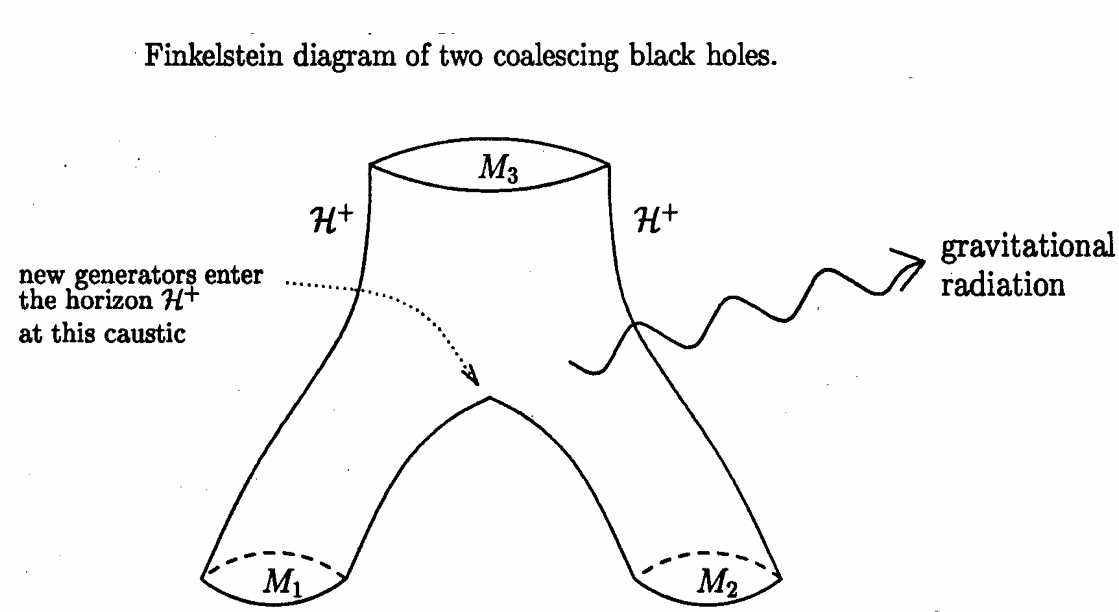

Fig. 7.1.

The case of Kerr BHs is slightly more complicated. We have to return to (6.1.), set $t = const.$ ($dt = 0$) and $r = r_+$ ($dr = 0$). Then the metric reads

$$ds^2 = (r_+^2 + a^2 \cos^2\theta)\, d\theta^2 + \frac{(r_S r_+)^2}{r_+^2 + a^2\cos^2\theta} \sin^2\theta\, d\phi^2 \qquad (7.8)$$

To calculate the area of the Kerr horizon we need the determinant of the metric tensor which has in (7.8) only two non vanishing diagonal elements, that is $g = g_{\theta\theta} \cdot g_{\phi\phi}$. Then the area integral becomes

$$A = \iint \sqrt{g}\, d\theta\, d\phi = \iint r_S\, r_+ \sin\theta\, d\theta\, d\phi \qquad (7.9)$$

$$A = 4\pi\, r_S \left[\frac{r_S}{2} + \left(\frac{r_S^2}{4} - a^2\right)^{1/2} \right] \qquad (7.10)$$

or with $r_S = 2M$ (remember we set $G = c = 1$)

$$A = 8\pi M\left[M + (M^2 - a^2)^{1/2}\right] \qquad (7.11)$$

The area is a maximum when the momentum vanishes $a \to 0$. The application of the area theorem makes it impossible to make a ***naked singularity*** just by adding particles and their angular momentum to a BH. From the differentiation of (7.8) we find amongst others two critical terms

$$\frac{M^2 \delta M}{\sqrt{M^2 - a^2}} \text{ and } \frac{-aM\delta a}{\sqrt{M^2 - a^2}} \qquad (7.12)$$

Since

$$\delta A > 0 \qquad (7.13)$$

should always hold it follows that $M\delta M > a\delta a$. Therefore we note oncemore that it is impossible to prepare a naked singularity in the above mentioned way. If it would be possible it would leave a BH without a horizon and would enable an observer to "look" into the singularity, a point which is causally decoupled from the otherwise predictable spacetime.

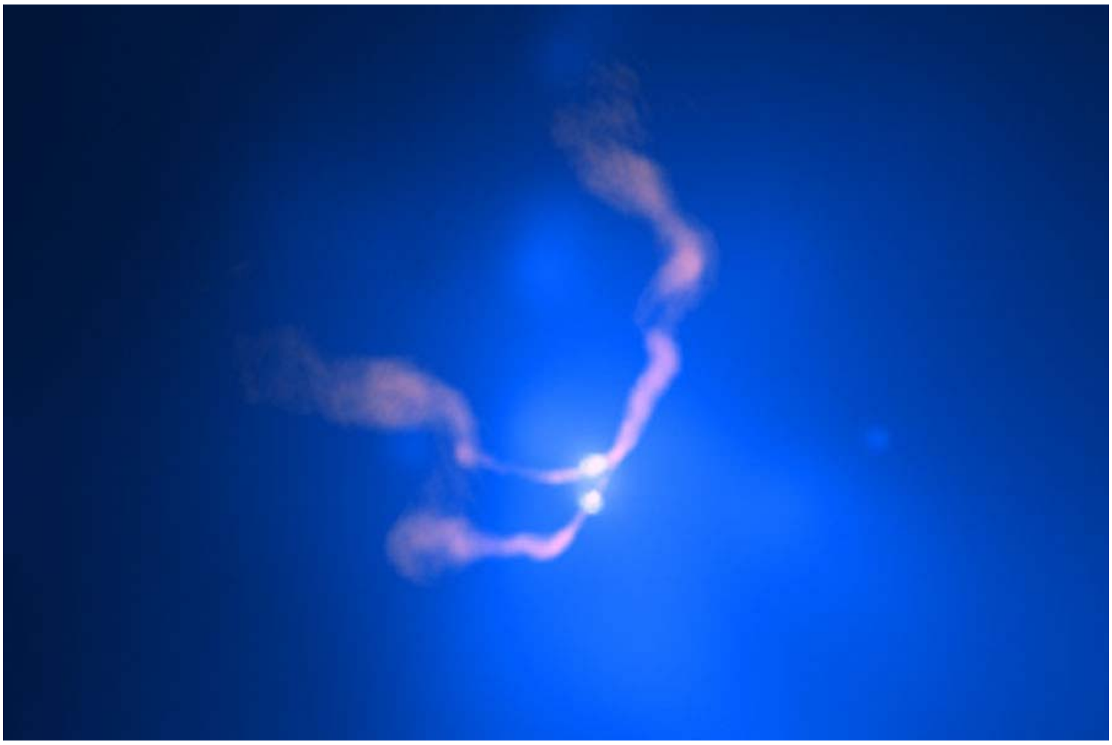

Fig. 7.2. This picture is a composite of X-ray and radiofrequency data. It shows 2 bright sources in the center of the elliptic galaxy 3C 75 interpreted as two SMBHs with jets. It can be expected that the BHs interact, their distance decreases with under emittance of gravitational radiation. In a future time they will merge and form one big SMBH. Credit: X-Ray: NASA / CXC / D. Hudson, T. Reiprich et al. (AIfA); Radio: NRAO / VLA/ NRL

Roger Penrose first formulated the ***Conjecture of Cosmic Censorship***. In a physical formulation we have the following statements (R.Wald 1984). First formulation:

The complete gravitational collapse of a body always results in a BH rather than a naked singularity; i.e., all singularities of gravitational collapse are hidden within BHs where they cannot be seen by distant observers.

Second (stronger) formulation:
All physically reasonable space-times are globally hyperbolic, i.e., apart from a possible initial singularity (such as the big bang singularity) no singularity is ever "visible" to an observer.

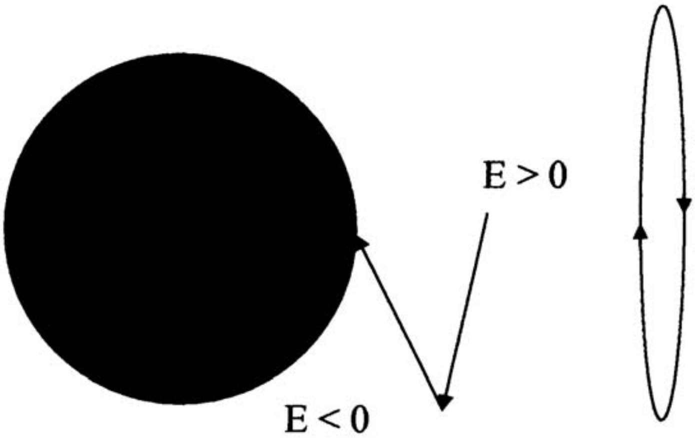

Fig. 7.3. Illustration of Hawking radiation: In this model pairs of particles are broken up in the gravitational field (strong curvature of space). The negative states disappear behind the horizon, the positive states are emitted as thermal radiation.

7.2. Hawking temperature and entropy

The universe is filled with cosmic radiation fields. In such environments a BH would continuously absorb radiation and would never come to equilibrium. Bekenstein (1973) and Hawking (1974, 1975) therefore investigated the thermodynamic of BHs. Hawking was able to show that in the presence of quantum fields a BH radiates with a thermal spectrum of temperature T

$$T = \frac{\kappa \hbar}{2\pi k_B} = \frac{\hbar}{8\pi k_B M} = 10^{-7}\, K \left[\frac{M_{sol}}{M}\right] \qquad (c = G = 1) \tag{7.14}$$

Since the area can only increase with time it should be proportional to the entropy $dA \propto dS$. From classical thermodynamics we have

$$dS = \frac{dq}{T} = \frac{dU + dW}{T} \tag{7.15}$$

If no work is applied to the system then $dW = 0$. Furthermore we set $dU = d(Mc^2)$ and write using (7.7)

$$dU = T\, dS \tag{7.16}$$

or

$$d(Mc^2) = \frac{1}{32\pi}\frac{dA}{M} = \frac{\kappa}{8\pi} dA \tag{7.17}$$

Comparing (7.17) with (7.14) we find for the entropy of a BH ($c = G = 1$)

$$S = \frac{1}{4}\frac{k_B}{\hbar}A \qquad (7.18)$$

A solar mass BH has an enormous entropy in units of Boltzmann's constant $S = 1{,}0 \cdot 10^{77}\, k_B$, whereas for the classical entropy of the sun we estimate $S_{sol} = 2 \cdot 10^{58}\, k_B$, this is 19 orders of magnitude less. The most general form of (7.17) is known as the ***first law of BH thermodynamics***

$$dM = T\,dS + \Phi_H\, dQ + \Omega_H\, dJ \qquad (7.19)$$

where Φ and Ω have to be taken at the horizon. The ***second law*** can be formulated as follows

$$\frac{dA}{dt} \geq 0 \qquad (7.20)$$

At the end of this section we will give once more the relevant quantities restoring the c's and G's

$$A = 4\pi\left(\frac{2GM}{c^2}\right)^2; \qquad S = \frac{k_B}{\hbar}\cdot\frac{A}{4} = \frac{k_B c^3}{G\hbar}\cdot\frac{1}{4}A; \qquad T = \frac{\kappa}{2\pi} = \frac{\hbar\, c^3}{8\pi\, k_B\, G M}$$

(7.21)

7.3. Decay of BHs.

If BHs radiate they must have a limited life time. Considering Hawking's development the radiation should be a thermal radiation obeying the Stefan-Boltzmann law

$$\frac{dE}{dt} = \sigma T^4 A \propto M^{-4} \cdot M^2 \propto M^{-2} \qquad (7.22)$$

which we can reformulate

$$\frac{dE}{dt} = \frac{dM}{dt} = \frac{M}{\tau} \qquad \text{or in terms of } \tau \qquad \tau = \frac{E}{dE/dt} \propto M^3 \qquad (7.23)$$

Putting in some numbers we get an estimate of the BHs life time under the assumption that neither matter nor radiation will be absorbed

$$\tau \approx \frac{M^3}{\hbar} \approx 10^{10} \cdot \left[\frac{M}{10^{12}\, kg}\right]^3 yr \qquad (7.24)$$

If there would exist primordial BHs left over from the big bang they must have enough mass to survive $13{,}7 \cdot 10^9\, yr$, the age of the universe, a discussion which is left to the problems.at the end of this lecture. There is in general no thermal equilibrium between BHs and their environment. Take as an example, a black hole of one solar mass. It has a temperature of only 60 nK (Nanokelvin). In fact, such a black hole would absorb far more cosmic microwave background radiation than it emits. A black hole of 4.5×10^{22} kg (about the mass of the moon) would be in equilibrium at 2.7 K, absorbing as much cosmic radiation as it emits. Yet smaller primordial BHs would emit more than they absorb and thereby lose mass.

7.4. Black Hole thermodynamics a road to quantum gravity?

We are now (I hope) convinced that a BH is a thermodynamic system. The thermodynamic expression TdS used in the last section is purely classical. However both quantities $S \propto 1/\hbar$ and $T \propto \hbar$ contain Planck's constant. This gives us a motivation to leave the classical description and consider a statistical interpretation of the BH's entropy. Remembering Boltzmann's entropy law, engraved on the stone of his tombe

$$S = k_B \ln w \qquad (7.25)$$

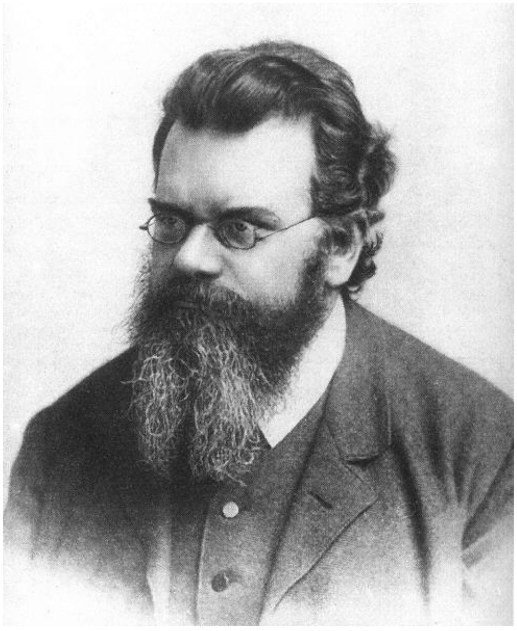

Fig. 7.4. Ludwig Eduard Boltzmann 1844 – 1906

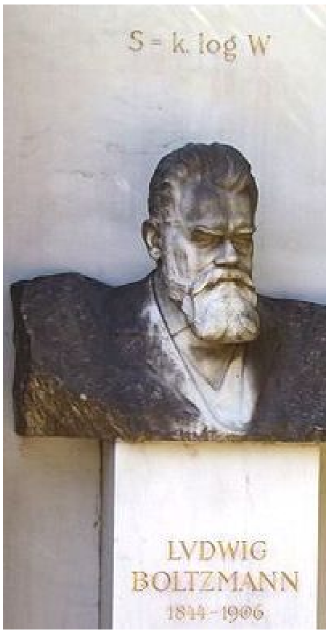

Fig. 7.5. The bust of Boltzmann and the engraved entropy formula on his grave in the Zentralfriedhof of Vienna

The entropy is the logarithm of *w* the number of independent states of the BH. But now we arise at the central question: what are these microstates of the BH? John A. Wheeler contributed early to this question with an intriguing suggestion. The entropy written in the denominator (7.18) contains the square of the Planck length

$$\frac{1}{4} \cdot \frac{A}{l_P^2} = \ln w \qquad (7.26)$$

with

$$l_P^2 = \frac{\hbar G}{c^3} = \left[1{,}616 \cdot 10^{-35} m\right]^2 \qquad (7.27)$$

This interpretation tells us that the statistical entropy is just proportional to the horizon area measured in Planck units. The prefactor ¼ is a result of Hawking's calculation and has no easy interpretation. But how does (7.26) lead to a new interpretation of the BH's entropy? The Planck length is the smallest physically reasonable length unit. Any discussion about quantum gravity would confront us with Planck length, Planck time and Planck energy. Wheeler went a step further and proposed to consider l_P^2 as smallest unit of information, i.e. ***he connected*** l_P^2 ***with one bit,*** with one either-or decision. The information, i.e. the number of bits is then

$$p = \frac{A_H}{l_P^2} \tag{7.28}$$

If the total number of states is $N = 2^P$ or

$$p = \log_2(N) = \ln N / \ln 2 \,. \tag{7.29}$$

So we may write for the entropy

$$S \propto \frac{1}{4} p = \frac{1}{4 \ln 2} \ln N \tag{7.30}$$

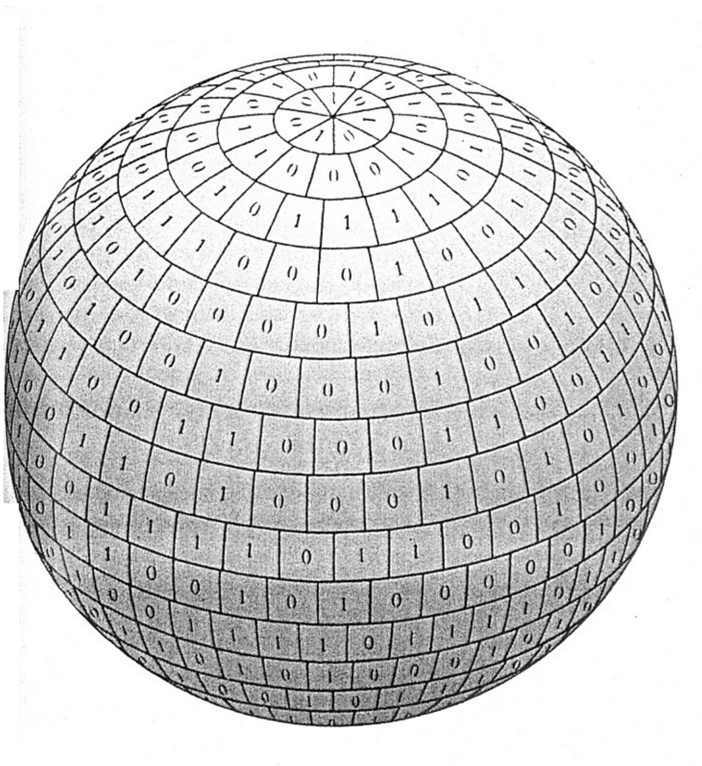

Fig. 7.6. The surface of a sphere divided in unit areas of one bit. Credit: Claus Kiefer.

Using fig. 7.6 we interpret the N states as the sum of all allowed combinations 01011001110101…….on the sphere. That is as far as we can go without a quantum theory of gravity.

There is at present unfortunately no general accepted theory of quantum gravity. The most developed approaches are string theory (ST) and loop quantum gravity (LQG). Instead of four dimension ST works in 10 or 11 dimensions but claime to be able to describe all four fundamental forces in one unified theory with nearly very few free parameter to be fixed by experiments. However to regain the 4-dimensional space-time of our empirical world the "extra dimensions" must be compactified to make them unobservable (high energy experiments were carefully checked for hints of higher dimension but up to now without success). There are many possible way of "compactification" (estimates give about 10^{500} possibilites) and as many different ways to find the ground state of the fields. That is one reason why some critics find ST useless since its "truth" cannot be checked by any application to the empirical world. LPG is less ambitious and does not attempt to unify all interactions. Instead it is a Hamiltonian theory of gravity with newly constructed canonically conjugate variables. The basic gravitational configuration variable is a SU(2) connection acting on a 3-manifold representing "space". The momenta are field variables with properties very similar to the electric field. Area and volume appear as operators. The area operator has eigenvalues which are multiples of l_P^2. Both ST and LQG have been applied to the BH horizon problem. With ST only extremal Kerr-Newman BHs have been treated. Hawking's entropy prefactor "1/4" could be verified, undeniably a great success.

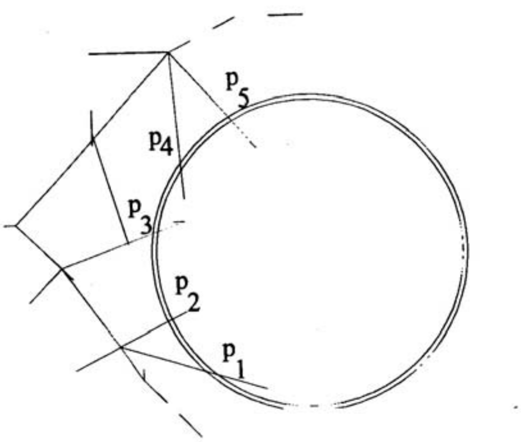

Fig. 7.7. In loop quatum gravity the "spins" of the space eigenstates pierce an isolated boundary (the horizon) which is originally flat. But at each puncture is a quantized deficit angle which adds up to the topology of the 2-sphere. Furthermore, the punctures are locations of elementary areas which carry information. Credit: A. Ashtekar.

LQG has been applied very generally to BHs in equilibrium. The solutions look like random spin lattices (which are excited space states) with an isolated horizon as inner boundary. For an exterior observer the isolated horizon is a physical boundary which separates the region accessible to him or her from the one which is not. The specific solutions pierce the boundary, endowing it with elementary areas. The maximum entropy is gained when the spin for each puncture is $m_n = \pm 1/2$. SU(2) invariance leads to a singlet state over the horizon surface

$$\sum_{n=1}^{p} m_n = 0 \qquad (7.31)$$

N states are then given and approximated (using Stirling's formula) for large p by

$$N(p) = \binom{p}{p/2} \cong 2^P \qquad (7.32)$$

where Stirling's approximation for very large „p" has been applied ($p! \approx p\ln p + \ldots$).
The result reproduces very nicely equ. (7.26). A more careful consideration reveals that triplets also contribute which leads to a correction of (7.32). The final result of the LQG approach with corrections gives

$$\ln N(p) = \frac{A_H}{4l_P^2} - \frac{3}{2}\ln\left(\frac{A_H}{4l_P^2}\right) - \frac{1}{2}\ln\left(\frac{\pi}{8(\ln 2)^3}\right) + \ldots \qquad (7.33)$$

Ans verifies in lowest order our result given above.

7.5. The holographic principle and the world as a hologram.

The entropy of a BH is extraordinary large, more than18 orders of magnitude larger than the entropy of an ideal gas of the same mass. Why is this so? To answer this question we first remember that statistical entropy is a measure of our ignorance, it is the information we just do not have (ignoring a factor of order unity). However, its magnitude depends on the coarse graining, on how much details we want to know. A good example offers the resolution of pictures taken by a digital camera. The resolution depends on the number of pixel per area. The more pixel per area the higher the resolution, the higher is the information content of the picture. In the BH entropy we find maximum resolution. This highest "pixel density" is a consequence of the choice of the Planck area $l_P^2 = G\hbar \cdot c^{-3} = (1{,}616 \cdot 10^{-35}\, m)^2$, likely the area element of quantum gravity.
However, this is not a free choice, but a consequence of the use of Hawking's temperature T_H. Hawking derived T_H (which contains $\hbar$) from an investigation of quantum fields in curved spacetime (i.g. Schwarzschild metric). With the use of the well known thermodynamic relation $dE = C \cdot TdA = TdS$ we found that the horizon area is given in units of l_P^2. Therefore dS contains G and $\hbar$ although no quantum gravity approach was applied. In the product of T and dS Planck's constant $\hbar$ as a factor compensates which makes $dE = TdS$ again a classical thermodynamic expression.

Thus BH entropy S_{BH} drives the resolution down to the Planck's area l_P^2, a unit expected to play a role in quantum gravity. Furthermore we find a ***maximum entropy bound*** for any mass and/or energy in a local environment of the universe if we demand

$$S(M) \leq S_{BH}(M) \qquad (7.34)$$

Equivalently we have for the maximum information content

$$p(M) \leq \frac{A_{BH}(M)}{4l_P^2} \qquad (7.35)$$

BHs are very simple objects, characterized by only 3 parameters: mass, angular momentum and charge. Since they are macroscopic objects they must be highly degenerate, i.e. include a large number of microstates as we have found

$$N(p) = 2^p = \exp\frac{S_{BH}}{k_B \ln 2} \qquad (7.36)$$

Even without a generally accepted theory of quantum gravity BH thermodynamics gives us some hints how such a theory would limit the number of microstates.

Another important and also very strange result is the reduction of spacial dimensions: the information from a volume of space is found on the surface area A_{BH} . The generalisation of this result is known under the name "***holographic principle***": All information from some region of space is contained on the boundary of this region as a "hologram".

This principle was recently applied even to cosmological horizons. In the last years the holographic principle helped to develop a new treatment of field theoretical problems. It has been found that the physics of gravity in 5-dimensional anti-de-Sitter spacetime (AdS) is equivalent to a certain Yang-Mills theory defined on the boundary of the AdS spacetime. This is considered to be a "hologram" of the physical world happening in 5 dimensions.

This strange correspondence may remind us of Plato's "Allegory of the Cave". It tells us of people chained in a cave who can follow the outside life only by the moving shadows projected with the outside light on the walls of their cave. This seems to be a rather pessimistic view of our cognition which seemed to Socrates (from whom Plato learned about the allegory) imperfect enough as to lose a full spatial dimension. In contrast, the conjecture of ***holographic principle*** offers a quite optimistic view at least as far as the world of physics is concerned: The information of a D-dimensional spacetime is already contained on a (D-1)-dimensional hypersurface. Do you like to know more about this exiting correspondence? You will find it in the nice little book of L. Susskind and J. Lindesay refered below.

7.6. Problems

7.6.1. Estimate a lower mass limit for primordial BHs which could have survived the cosmic evolution (age of the universe $14{,}7 \cdot 10^9$ years).

7.6.2. Calculate the entropy of the sun. Assume that the sun consists of completely ionized hydrogen gas (solar mass $1{,}99 \cdot 10^{30}$ kg).

7.6.3. The radius if the horizon of the observable universe was about $400\ 000\, ly$ when radiation decoupled from matter. Give an estimate of the maximal information content in this early cosmos (1 Ly = $9{,}46 \cdot 10^{15}$ meter). There remains a problem: The information in the universe cannot change. However the radius of the horizon grows with the expansion. How can this apparent contradiction be resolved?

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