Integrability in QCD and beyond

Vladimir M. Braun
University of Regensburg

thanks to

Sergey Derkachov, Gregory Korchemsky and Alexander Manashov
Regensburg
Outline

In this talk:

- Renormalization in conformally-covariant form
- Integrability on the example of three-quark operators in QCD
- Violation of integrability: Mass gaps
- SUSY extensions: From $\mathcal{N} = 0$ to $\mathcal{N} = 4$
- Further QCD example: Polarized deep-inelastic scattering

not included:

- *High energy scattering in QCD*
- *Full $\mathcal{N} = 4$ SUSY and gauge/string correspondence*
1926: Heisenberg

$$\mathbb{H}_{s=1/2} = -\sum_{n=1}^{L} \left( \hat{S}_n \cdot \hat{S}_{n+1} - \frac{1}{4} \right)$$

1981-83: Generalizations for arbitrary spin

$$\mathbb{H}_s = \sum_{n=1}^{L} H(J_{n,n+1})$$

$$J_{n,n+1}(J_{n,n+1} + 1) = (\hat{S}_n + \hat{S}_{n+1})^2$$

Complete integrability

Number of degrees of freedom = number of conservation laws

- Conceptually interesting
- Powerfull machinery:
  
  **Algebraic Bethe Ansatz (ABA)**
  
  **Method of separation of Variables (SoV)**
at the same time, in QCD . . .

1973-74: Anomalous dimensions of leading twist operators

$$\gamma_N = C_F \left( -3 - \frac{2}{(N+1)(N+2)} + 4\psi(N+2) - 4\psi(1) \right)$$

1976-78: High-energy (Regge) asymptotics of scattering amplitudes

$$E_{N,\nu}^{BFKL} = 2\psi(1) - \psi \left( \frac{n+1}{2} + i\nu \right) - \psi \left( \frac{n+1}{2} - i\nu \right)$$

1991: Anomalous dimensions of quark-antiquark-gluon operators, $N_c \to \infty$

Exact analytic solution

Hidden symmetry?
Example:

**Twist-three operators** \( \Leftrightarrow \)** Leading-twist three-particle parton distributions

E.g. baryon distribution amplitudes \( B = N, \Delta, \ldots \)

\[
\langle 0 | q(z_1)q(z_2)q(z_3) | B(p, \lambda) \rangle = \ldots \int_0^1 dx_1 dx_2 dx_3 \delta(\sum x_i - 1) e^{-ip(x_1 z_1 + x_2 z_2 + x_3 z_3)} \phi_B(x_i, \mu^2)
\]

- \( q^\uparrow q^\uparrow q^\uparrow \Rightarrow \phi_\Delta^{\lambda=3/2}(x_i, \mu^2), \)
- \( q^\uparrow q^\downarrow q^\uparrow \Rightarrow \begin{cases} \phi_N^{\lambda=1/2}(x_i, \mu^2) \\ \phi_\Delta^{\lambda=1/2}(x_i, \mu^2) \end{cases} \)

- quark fields “live” on a light-ray \( z^2 = 0 \)
Problem: Proliferation of degrees of freedom

Moments of distribution amplitudes $\Leftrightarrow$ local operators:

$$\varphi(x_i) \rightarrow \varphi(k_i) = \int \mathcal{D} x_i x_1^{k_1} x_2^{k_2} x_3^{k_3} \varphi(x_i, \mu^2)$$

$$q(z_1) q(z_2) q(z_3) \rightarrow (D_+^k q) (D_+^k q) (D_+^k q)$$

Mixing matrix:

$$N = k_1 + k_2 + k_3$$

Rich spectrum of anomalous dimensions reflects complexity of genuine degrees of freedom

Example: $qqq$

$\gamma_{n=0,1,\ldots}^{N=14}$

$\lambda = 3/2$
Conformal Symmetry on the Light-Cone: $SL(2, \mathbb{R})$

$z \rightarrow z' = \frac{az + b}{cz + d}$, \quad ad - bc = 1 \quad \Phi(z) \rightarrow \Phi'(z) = \Phi \left( \frac{az + b}{cz + d} \right) \cdot (cz + d)^{-2j_{\Phi}}$

\[ j_q = 1, \quad j_g = \frac{3}{2} \text{ is conformal spin of the field} \]

\[
\begin{align*}
L_- \Phi(z) &= -\frac{d}{dz} \Phi(z) \\
L_+ \Phi(z) &= \left( z^2 \frac{d}{dz} + 2 j_{\Phi} z \right) \Phi(z) \\
L_0 \Phi(z) &= \left( z \frac{d}{dz} + j_{\Phi} \right) \Phi(z)
\end{align*}
\]

Casimir operators

\[ L^2 = L_0^2 - L_0 + L_+ L_- \quad L^2 \Phi(z) = j(j-1) \Phi(z) \]

Summation of spins

\[ L_{ik}^2 = \sum_{\alpha=0,1,2} (L_{i,\alpha} + L_{k,\alpha})^2 \quad L_{123}^2 = \sum_{\alpha=0,1,2} (L_{1,\alpha} + L_{2,\alpha} + L_{3,\alpha})^2 \]

- second-order differential operators on the space of $(z_1, z_2, z_3)$
RG equations in $SL(2)$-covariant form

Light-ray operators

\[
\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right\} B = \mathcal{H} \cdot B, \quad B(z_1, z_2, z_3) \simeq q(z_1)q(z_2)q(z_3)
\]

Two-particle structure:

\[
\mathcal{H}_{qq} = \mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}
\]

Renormalization = Displacement: Balitsky, V.B. 88'

\[
\mathcal{H}_{12} B(z_1, z_2, z_3) = \int \mathcal{D}\alpha \, \omega(\alpha_1 \alpha_2) \, B(\bar{\alpha}_1 z_1 + \alpha_1 z_2, \bar{\alpha}_2 z_2 + \alpha_2 z_1, z_3)
\]

where \[
\int \mathcal{D}\alpha = \int d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3), \quad \bar{\alpha} = 1 - \alpha
\]
Conformal symmetry $\rightarrow [L_{\pm,0}, \mathcal{H}_{12}] = 0$

$$\omega(\alpha_1 \alpha_2) = \tilde{\alpha}^2 j_1 - 2 \tilde{\alpha}^2 j_2 - 2 \varphi \left( \frac{\alpha_1 \alpha_2}{\tilde{\alpha}_1 \tilde{\alpha}_2} \right)$$

$\tilde{\alpha} = 1 - \alpha$

Explicit calculation

$\Rightarrow \varphi(x) = \delta(x),$

$\Rightarrow \varphi(x) = 1$
RG equations in $SL(2)$-covariant form — Hamiltonian approach

write $\mathcal{H}$ as a function of Casimir operators

$$L_{ik}^2 = -\frac{\partial}{\partial z_i} \frac{\partial}{\partial z_k} (z_i - z_k)^2 \equiv \hat{J}_{ik}(\hat{J}_{ik} - 1)$$

$$\mathcal{H}^{\nu}_{12} = 2[\psi(J_{12}) - \psi(2)]$$

$$\mathcal{H}^\lambda_{qqq} = \mathcal{H}^\nu_{12} + \mathcal{H}^\nu_{23} + \mathcal{H}^\nu_{13}$$

$$\mathcal{H}^{\lambda=3/2}_{qqq} = \mathcal{H}^\nu_{qqq} - \mathcal{H}^e_{12} - \mathcal{H}^e_{23}$$

$$\mathcal{H}^{\lambda=1/2}_{qqq} = \mathcal{H}^{\lambda=3/2}_{qqq} - \mathcal{H}^e_{12} - \mathcal{H}^e_{23}$$

Have to solve $\mathcal{H}\Psi_{N,n} = \mathcal{E}_{N,n}\Psi_{N,n}$ — A Schrödinger equation with Hamiltonian $\mathcal{H}$
going over to local operators:

In helicity basis (BFKL 85') obtain a
HERMITIAN Hamiltonian
with respect to the conformal scalar product

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_0^1 dx_i \delta\left( \sum x_i - 1 \right) x_1^{2j_1-1} x_2^{2j_2-1} x_3^{2j_3-1} \Psi_1^*(x_i) \Psi_2(x_i)
\]

that is, conformal symmetry allows for the transformation of the mixing matrix:

\[
\begin{pmatrix}
\text{forward} \\
\text{non-hermitian}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\text{non-forward} \\
\text{hermitian}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\text{hermitian} \\
\text{in conformal basis}
\end{pmatrix}
\]

\[
\begin{pmatrix}
N + 1 \\
N(N + 1)/2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
N + 1
\end{pmatrix}
\]
Hamiltonian approach — continued (2)

- Eigenvalues (anomalous dimensions) are real numbers
- “Conservation of probability” – possibility of meaningful approximations to exact evolution
- Systematic $1/N_c$ expansion

\[ \mathcal{E}_{N,n} = N_c E_{N,n} + N_c^{-1} \delta E_{N,n} + \ldots \]
\[ \Psi_{N,n} = \Psi_{N,n}^{(0)} + N_c^{-2} \delta \Psi_{N,n} + \ldots \]

with the usual quantum-mechanical expressions

\[ \delta E_{N,n} = \| \Psi_{N,n}^{(0)} \|^{-2} \langle \Psi_{N,n}^{(0)} | \mathcal{H}^{(1)} | \Psi_{N,n}^{(0)} \rangle \]

etc.
Complete Integrability

- Conformal symmetry implies existence of two conserved quantities:

\[ [\mathcal{H}, L^2] = [\mathcal{H}, L_0] = 0 \]

- For \( qqq \) and \( GGG \) states with maximum helicity and for \( qGq \) states at \( N_c \to \infty \) there exists an additional conserved charge

\[
\begin{align*}
\mathcal{H} \Psi &= \mathcal{E} \Psi \\
Q \Psi &= q \Psi
\end{align*}
\]

\[ \Rightarrow \quad \mathcal{E} = \mathcal{E}(q) \]

- Anomalous dimensions can be classified by values of the charge \( Q \):
WKB expansion of the Baxter equation $\Rightarrow 1/N$ expansion

- Equation $Q\Psi = q\Psi$ is much simpler as $\mathcal{H}\Psi = \mathcal{E}\Psi$
- Non-integrable corrections are suppressed at large $N$
- Can use some techniques of integrable models

$$\mathcal{E}(N, \ell) = 6 \ln \eta - 3 \ln 3 - 6 + 6\gamma_E - \frac{3}{\eta} (2\ell + 1) - \frac{1}{\eta^2} (5\ell^2 + 5\ell - 7/6) - \frac{1}{72\eta^3} (464\ell^3 + 696\ell^2 - 802\ell - 517) + \ldots$$

$$\eta = \sqrt{(N+3)(N+2)}$$

An integer $\ell$ numerates the trajectories:
(-semiclassically quantized solitons: Korchemsky, Krichever, ’97)

- Integrability imposes a nontrivial analytic structure
The ‘dispersion curve’ $E(q)$

\[ E(q) = 2 \ln 2 - 6 + 6 \gamma_E + \]
\[ + 2 \text{Re} \sum_{k=1}^{3} \psi(1 + i \eta^3 \delta_k) + \mathcal{O}(\eta^{-6}) \]

$\delta_k$ are defined as roots of the cubic equation:

\[ 2\delta_k^3 - \delta_k - q/\eta^3 = 0 \]

Example:

$qqq$

$\lambda = 3/2$

$N = 30$

\[ \eta = \sqrt{(N+3)(N+2)} \]
WKB expansion of the Baxter equation \( \Rightarrow 1/N \) expansion — continued (2)

\[ \mathcal{E}(N, q) = \mathcal{E}(N, -q) \]

\[ q(N, \ell) = -q(N, N - \ell) \]

**Double degeneracy:**

**Example:**

\[ \lambda = \frac{3}{2} \]

BDKM '98
The ground state solution \((q = 0)\) for the wave function of \(\Delta^{3/2}\)

For even \(N\) \(q = 0\) is the solution with lowest energy

\[
x_1x_2x_3 \Psi_{N, q=0}(x_i) = x_1(1-x_1)C_{N+1}^{3/2}(1-2x_1) + x_2(1-x_2)C_{N+1}^{3/2}(1-2x_2) + x_3(1-x_3)C_{N+1}^{3/2}(1-2x_3)
\]

The lowest anomalous dimension

\[
\mathcal{E}(N, q = 0) = 4\Psi(N + 3) + 4\gamma_E - 6
\]

Here \(x_1, x_2, x_3\) are fractions of the \(\Delta^{3/2}\) momentum carried by the three quarks
Breakdown of integrability: Mass gaps and bound states — "experiment"

Simplest case:

Difference between $\Delta^{\lambda=3/2}$ and $N(\Delta)^{\lambda=1/2}$

$$\mathcal{H}(\varepsilon) = \mathcal{H}_{3/2} - \varepsilon \left( \frac{1}{L_{12}^2} + \frac{1}{L_{23}^2} \right)$$

$$\mathcal{H}(\varepsilon = 1) = \mathcal{H}_{1/2}$$

Flow of energy levels
(anomalous dimensions):
Breakdown of integrability: Mass gaps and bound states — "theory"

In lower part of the $qqq$ spectrum

‘Perturbation’ $\langle q'| \frac{1}{L_{12}^2} + \frac{1}{L_{23}^2} | q \rangle \sim \frac{1}{\ln N}$

Level splitting $\sim \frac{1}{\ln^2 N}$

$\langle q'| \frac{1}{L_{12}^2} + \frac{1}{L_{23}^2} | q \rangle \sim \frac{1}{\ln N} \cos(\theta_q - \theta_{q'})$

Of order $\sim \ln N$ lower levels have to be rediagonalized

Phases of the cyclic permutations:

Matrix $(\ln N \times \ln N)$:

$$\frac{1}{\ln N} \begin{pmatrix} 1 & -1/2 & -1/2 & 1 & \ldots \\ -1/2 & 1 & -1/2 & -1/2 & \ldots \\ -1/2 & -1/2 & 1 & -1/2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{\text{diag}} \begin{pmatrix} \ln N & 0 & 0 & 0 & \ldots \\ 0 & \ln N & 0 & 0 & \ldots \\ 0 & 0 & \ln N & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\theta_q$

! A mass gap!
Breakdown of integrability: Mass gaps and bound states — "interpretation"

\[ \Delta^{\lambda=3/2} \text{ wave function} \]

\[ \varphi_{\Delta^{3/2}}(x_i)^\mu = \sum_{N=0}^{\infty} \varphi_{N,n=0}^\mu \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_{N,n=0}} \left\{ x_1(1-x_1)C_{N+1}^{3/2}(1-2x_1) + x_2(1-x_2)C_{N+1}^{3/2}(1-2x_2) + x_3(1-x_3)C_{N+1}^{3/2}(1-2x_3) \right\} \]

\[ \varphi_{\Delta^{1/2}}(x_i)^\mu = x_1x_2x_3 \sum_{N=0}^{\infty} \varphi_{N,n=0}^\mu \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_{N,n=0}} \left\{ P_N^{(1,3)}(1-2x_3) \pm P_N^{(1,3)}(1-2x_1) \right\} \]

Free motion

Scalar diquarks?
Supersymmetry: from $\mathcal{N} = 0$ to $\mathcal{N} = 4$

A. Belitsky, S. Derkachov G. Korchemsky and A. Manashov [hep-th/0403085]

- One expects that SUSY enhances integrability as one goes from $\mathcal{N} = 0 \rightarrow \mathcal{N} = 4$
- $QCD$-like integrability is extended and new features may appear
- Need a universal approach to treat simultaneously the operator mixing in various $\mathcal{N}$—extended SYM

### The light-cone superspace formalism

- **Quantize SYM in the light-cone gauge** $A_+(x) = 0$
- **Decompose fermions and gauge fields into** “good” $(\Psi_+, A_\perp)$ and “bad” $(\Psi_-, A_-)$ components
- **Integrate out nonpropagating (“bad”) fields and rewrite the action in terms of propagating (“good”) fields**
  $S_N = S_N(\Psi_+, A_\perp)$
- **Combine “good” fields into light-cone superfields** $\Phi(x, \theta^A)$

---

Kogut, Soper’70
\[ \mathcal{N} = 4 \text{ light-cone SYM} \]

**Light-cone superfield formulation of \( \mathcal{N} = 4 \text{ SYM} \)**

\[
S_{\mathcal{N}=4} = \int d^4 x d^4 \theta d^4 \bar{\theta} \left\{ \frac{1}{2} \Phi^a \Box \Phi^a - \frac{2}{3} g f^{abc} \left( \frac{1}{\partial_+} \Phi^a \Phi^b \partial \Phi^c + \frac{1}{\partial_+} \Phi^a \bar{\Phi}^b \partial \bar{\Phi}^c \right) \right. \\
\left. - \frac{1}{2} g^2 f^{abc} f^{ade} \left( \frac{1}{\partial_+} (\Phi^b \partial_+ \Phi^c) \frac{1}{\partial_+} (\bar{\Phi}^d \partial_+ \bar{\Phi}^e) + \frac{1}{2} \Phi^b \bar{\Phi}^c \Phi^d \bar{\Phi}^e \right) \right\}
\]

**Complex scalar \( \mathcal{N} = 4 \text{ chiral superfield} \)**

\[
\Phi = \partial^+_1 A + \theta^A \partial^+_1 \bar{\lambda}_A + \frac{i}{2!} \theta^A \theta^B \bar{\phi}_{AB} - \frac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \lambda^D - \frac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}
\]

- Chiral superfield spans all helicities of the particles in the supermultiplet
- The anti-chiral superfield is not independent
- All single-trace light-cone operators are built from a single chiral superfield

\[
O(Z_1, \ldots, Z_L) = \text{tr}\{\Phi(z_1, \theta_1) \ldots \Phi(z_L, \theta_L)\}
\]
... and back from $\mathcal{N} = 4$ to $\mathcal{N} = 0$

"Method of truncation": $S_{\mathcal{N}=4} \rightarrow S_{\mathcal{N}=2} \rightarrow S_{\mathcal{N}=1} \rightarrow S_{\mathcal{N}=0}$

$\Rightarrow \mathcal{N} = 2$ light-cone chiral superfield

$$\Phi(x, \theta) = \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \epsilon_{AB} \theta^A \theta^B \bar{\phi}(x)$$

$\Rightarrow \mathcal{N} = 1$ light-cone chiral superfield

$$\Phi(x, \theta) = \partial_+^{-1} A(x) + \theta \partial_+^{-1} \bar{\lambda}_A(x)$$

$\Rightarrow \mathcal{N} = 0$ light-cone chiral (super)field

$$\Phi(x) = \partial_+^{-1} A(x)$$

- chiral superfield $\Phi$ and its conjugate $\bar{\Phi}$ are independent
- Single-trace light-cone operators can be built from two superfields

**Integrable sector**

$$O_{\text{chiral}}(Z_1, \ldots, Z_L) = \text{Tr}\{\Phi(Z_1) \ldots \Phi(Z_L)\}$$

**Non-integrable sector**

$$O_{\text{mixed}}(Z_1, \ldots, Z_L) = \text{Tr}\{\Phi(Z_1) \ldots \bar{\Phi}(Z_L)\}$$
One-loop dilatation operator in chiral sector

A two particle kernel

\[ \mathcal{H}_{k,k+1} = \begin{array}{c}
\quad k \\
\quad k+1 \\
\end{array} \]

\[ \mathcal{V}_{k,k+1} (1 - \Pi_{k,k+1}) \]

acts as a displacement in the light-cone superspace

\[ Z = (z, \theta^A) \quad (A = 1, \ldots, N) \]

\[ \mathcal{V}_{12} \mathcal{O}(Z_1, Z_2) = \int_0^1 \frac{d\alpha}{(1-\alpha)\alpha^2} \left[ 2\alpha^2 \mathcal{O}(Z_1, Z_2) - \mathcal{O}(\alpha Z_1 + (1-\alpha) Z_2, Z_2) - \mathcal{O}(Z_1, \alpha Z_2 + (1-\alpha) Z_1) \right] \]

- Projection operator, \( (\Pi_{12})^2 = \Pi_{12} \)

\[ \Pi_{12} \mathcal{O}(Z_1, Z_2) = \frac{1}{2} (1 + Z_{12} \cdot \partial_Z) \mathcal{O}(Z, Z_2) \bigg|_{Z=Z_2} + \frac{1}{2} (1 + Z_{21} \cdot \partial_Z) \mathcal{O}(Z_1, Z) \bigg|_{Z=Z_1} \]

Compare this with QCD expression for \( B(z_1, z_2, z_3) = q^\dagger(z_1) q^\dagger(z_2) q^\dagger(z_3) \): 

\[ \mathcal{H}_{12} B(z_1, z_2, \ldots) = \int_0^1 \frac{\alpha d\alpha}{1-\alpha} \left[ 2B(z_1, z_2) - B(\alpha z_1 + (1-\alpha) z_2, z_2) - B(z_1, \alpha z_2 + (1-\alpha) z_1) \right] \]

[For quarks \( j_q = 1 \), for chiral superfield \( j_\Phi = -1/2 \), hence different power of \( \alpha \)]
Dilatation operator on the light-cone superspace $Z = (z, \theta)$

- **The one-loop dilatation operator has the same, universal form for** $\mathcal{N} = 0, \mathcal{N} = 1,$ $\mathcal{N} = 2$ **and** $\mathcal{N} = 4$ SYM
  - *Dilatation operator displaces superfields in the superspace* $Z = (z, \theta^A)$
  - *The number of supercharges* $\mathcal{N}$ **defines the dimension of the superspace**
  - *Natural generalization of the QCD formulae* $Z = (z, \theta^A = 0)$

- **Two reductions of the** $\mathcal{N} = 4$ **dilatation operator**
  - *Expand in even light-cone coordinates, project on* $\theta = 0$
    - **Maximal helicity gluon operators**
      - $\mathbb{H} = SL(2; R)$ magnet
      - [Braun, Derkachov, Manashov; Belitsky; G.K '98]
  - *Expand in odd $\theta$-coordinates, project on* $z_i = 0$
    - **BMN-like operators:** $O_{\text{scal}} = \text{Tr}\{\phi_{A_1 B_1}(0) \cdots \phi_{A_L B_L}(0)\}$
      - $\mathbb{H} = SO(6)$ magnet
      - [Minahan, Zarembo '02]

- **In general:** collinear subgroup of the superconformal symmetry group:
  *Heisenberg (super) spin magnet with the* $SL(2|\mathcal{N})$ **symmetry**
The case of $\mathcal{N} = 4$ SYM is still special:

- All light cone operators belong to the $SL(2|\mathcal{N})$ sector since $\bar{\Phi}(Z)$ is not dynamically independent.

- Integrability extends to all operators (including those with “bad” components) and the collinear subgroup $SL(2|\mathcal{N})$ extends to the full superconformal symmetry group $PSU(2, 2, \mathcal{N})$.

[Beisert, Staudacher, ’03]

Why?

- Crucial: Superconformal symmetry relates operators of different twist.
- Superconformal transformations applied to light-cone operators $\phi^i D^N_\pm \phi^i$ yield combinations of operators of different twist but having the same anomalous dimension.
- This is not the case in QCD where operators of different twists are believed to be dynamically independent.

- Probably persists to all loops.
Outlook

- **One-loop dilatation operator (evolution equations) in QCD is/are integrable in some sectors**
  - **Classical bremsstrahlung:**
    \[ A \sim \frac{d\omega}{\omega} \frac{d\theta}{\theta} \]
  - **Cusp anomalous dimension:** for \( N \to \infty \)
    \[ \gamma_N \sim \Gamma_{\text{cusp}}(\alpha_s) \log N \]
  - **Integrability appears as a consequence of the existence of massless vector particles**

- **Offers powerful machinery**

- **Open problems in QCD context:**
  - **Analytic structure of the spectrum — Parton interpretation in higher twists**
  - **From \( \mathcal{N} = 4 \) to \( \mathcal{N} = 0 \): Rethinking of the role of non-quasipartonic operators; properties of anomalous dimensions in all twists**
  - **Beyond one loop: Formal conformal limit**
    \[ A = A^{\text{conformal}} + \frac{\beta(g)}{g} \Delta A \]
    *Integrability? Particular models?*
  - **Breaking of integrability vs. breaking of conformal symmetry: physics issues?**
Addendum: Polarized deep-inelastic scattering

Cross section:

\[
\frac{d^2\sigma}{d\Omega dE_{l'}}(\uparrow\uparrow\downarrow) = \frac{8\alpha^2 E_{l'}}{Q^4\nu} \left[ F_2(x, Q^2) \cos^2 \frac{\theta}{2} + \frac{2\nu}{M} F_1(x, Q^2) \sin^2 \frac{\theta}{2} \right],
\]

\[
\frac{d^2\sigma}{d\Omega dE_{l'}}(\uparrow\uparrow\downarrow) = \frac{8\alpha^2 E_{l'}}{Q^4} x \left[ g_1(x, Q^2) \left( 1 + \frac{E_{l'}}{E_l} \cos \theta \right) - \frac{2Mx}{E_l} g_2(x, Q^2) \right]
\]

At tree level:

\[
g_1(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p, s_z | \bar{q}(0) \gamma_5 q(\lambda n) | p, s_z \rangle
\]

\[
g_T(x) = \frac{1}{2M} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p, s_{\perp} | \bar{q}(0) \gamma_{\perp} \gamma_5 q(\lambda n) | p, s_{\perp} \rangle
\]

— quark distributions in longitudinally and transversely polarized nucleon, respectively
Polarized deep-inelastic scattering — continued

Beyond the tree level

\[
\langle N(p, s)|\bar{q}(z_1)G(z_2)q(z_3)|N(p, \lambda)\rangle =
\]

\[
= \ldots \int_{-1}^{1} dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3) e^{-ip(x_1 z_1 + x_2 z_2 + x_3 z_3)} D_q(x_i, \mu^2)
\]

\[
\bar{q} G q \quad \Rightarrow \quad g_2^{p-n}(x_B, \mu^2)
\]

\[
\bar{q} G q \quad \Rightarrow \quad g_2^{p+n}(x_B, \mu^2)
\]

quark-gluon correlations in the nucleon
Renormalization of quark-gluon operators, flavor non-singlet

\[
\mathcal{H}_{qGq} = N_c \mathcal{H}^{(0)} - \frac{2}{N_c} \mathcal{H}^{(1)},
\]

\[
\mathcal{H}^{(0)} = V_{qq}^{(0)} (J_{12}) + U_{qq}^{(0)} (J_{23}),
\]

\[
\mathcal{H}^{(1)} = V_{qq}^{(1)} (J_{12}) + U_{qq}^{(1)} (J_{23}) + U_{qq}^{(1)} (J_{13}).
\]

where

\[
V_{qq}^{(0)} (J) = \psi(J + 3/2) + \psi(J - 3/2) - 2\psi(1) - 3/4,
\]

\[
U_{qq}^{(0)} (J) = \psi(J + 1/2) + \psi(J - 1/2) - 2\psi(1) - 3/4,
\]

\[
V_{qq}^{(1)} (J) = \frac{(-1)^{J-5/2}}{(J - 3/2)(J - 1/2)(J + 1/2)},
\]

\[
U_{qq}^{(1)} (J) = -\frac{(-1)^{J-5/2}}{2(J - 1/2)},
\]

\[
V_{qq}^{(1)} (J) = \psi(J) - \psi(1) - 3/4,
\]

\[
U_{qq}^{(1)} (J) = \frac{1}{2} [\psi(J - 1) + \psi(J + 1)] - \psi(1) - 3/4.
\]
Open inhomogeneous spin chain

The lowest level is special and is separated from the rest of the spectrum by a “mass gap”

This special level was found by ABH 91’ and it determines the evolution of $g_2(x, Q^2)$ to the leading logarithmic accuracy.

It corresponds to the particular combination of quark-antiquark-gluon operators that can be reduced to the quark-antiquark twist-three operator using equations of motion.

Detailed study:

Belitsky; Derkachov, Korchemsky, Manashov ’00
The highest $qGq$ level is separated from the rest by a finite gap and is almost degenerate with the lowest $GGG$ state.

$qGq$ levels: crosses

$GGG$ levels: open circles

- Interacting open and closed spin chains
Flavor singlet $qGq$ and $GGG$ operators: WKB expansion

Energies: $j = N + 3$

\[
\mathcal{E}_q^{\text{low}} = N_c \left[ 2\psi[j] + \frac{1}{j} - \frac{1}{2} + 2\gamma_E \right] - \frac{2}{N_c} \left[ \ln j + \gamma_E + \frac{3}{4} - \frac{\pi^2}{6} + \ldots \right] \\
\mathcal{E}_q^{\text{high}} = N_c \left[ 4\ln j + 4\gamma_E - \frac{19}{6} - \frac{19}{3j^2} \right] + \frac{2}{3} n_f + \mathcal{O}(1/j^3) \\
\mathcal{E}_G^{\text{low}} = N_c \left[ 4\ln j + 4\gamma_E + \frac{1}{3} - \frac{\pi^2}{3} + \frac{1}{j} \left( \frac{2\pi^2-18}{3} (\ln j + \gamma_E) - \frac{\pi^2}{3} \right) \right] + \mathcal{O}(\ln^2 j/j^2)
\]

To the leading logarithmic accuracy, the structure function $g_2(x, Q^2)$ is expressed through the quark distribution

\[
g_2(x, Q^2) = g_{WW}^{WW}(x, Q^2) + \frac{1}{2} \sum_q e_q^2 \int_x^1 \frac{dy}{y} \Delta q_T^+(y, Q^2)
\]
Approximate evolution equation for the structure function $g_2(x, Q^2)$

Introduce flavor-singlet quark and gluon transverse spin distributions

\[
Q^2 \frac{d}{dQ^2} \Delta q_{T}^{+,S}(x; Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} \left[ P_{qq}^T(x/y) \Delta q_{T}^{+,S}(y; Q^2) + P_{qg}^T(x/y) \Delta g_T(y; Q^2) \right]
\]

\[
Q^2 \frac{d}{dQ^2} \Delta g_T(x; Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} P_{gg}^T(x/y) \Delta g_T(y; Q^2)
\]

with the splitting functions

\[
P_{qq}^T(x) = \left[ \frac{4C_F}{1-x} \right]_+ + \delta(1-x) \left[ C_F + \frac{1}{N_c} \left( 2 - \frac{\pi^2}{3} \right) \right] - 2C_F,
\]

\[
P_{qg}^T(x) = \left[ \frac{4N_c}{1-x} \right]_+ + \delta(1-x) \left[ N_c \left( \frac{\pi^2}{3} - \frac{1}{3} \right) - \frac{2}{3} n_f \right]
\]

\[
+ N_c \left( \frac{\pi^2}{3} - 2 \right) + N_c \ln \frac{1-x}{x} \left( \frac{2\pi^2}{3} - 6 \right),
\]

\[
P_{qg}^T(x) = -4n_f \left[ x - 2(1-x)^2 \ln(1-x) \right].
\]