Schwinger Model Simulations with Dynamical Overlap Fermions

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Abstract:

We present simulation results for the 2-flavour Schwinger model with dynamical overlap fermions. In particular we apply the overlap hypercube operator at seven light fermion masses.

We collect sizable statistics in the topological sectors 0 and 1. From the eigenvalue densities we evaluate the chiral condensate $\Sigma$ based on Random Matrix Theory in the $\epsilon$-regime.

For very light fermion masses our results for $\Sigma$ are close to analytical low energy predictions in the continuum.
**Schwinger model (QED$_2$)**: a popular toy model for QCD

\[
\mathcal{L}(\bar{\Psi}, \Psi, A_\mu) = \bar{\Psi}(x) \left[ \gamma_\mu (i\partial_\mu + gA_\mu) + m \right] \Psi(x) + \frac{1}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) .
\]

No spontaneous chiral symmetry breaking. **Chiral condensate**

\[
\Sigma(m) \equiv -\langle \bar{\Psi} \Psi \rangle \propto \left( \frac{m^{N_f - 1}}{\beta} \right)^{1/(N_f + 1)}
\]

$N_f$ : number of flavours ; $m$ : degenerate fermion mass ; $\beta = 1/g^2$

We consider $N_f = 2$. Low energy predictions for light fermions $m \ll 1/\sqrt{\beta}$

\[
\Sigma(m) = \begin{cases} 
0.372 \ (m/\beta)^{1/3} & [1] \\
0.388 \ (m/\beta)^{1/3} & [2] 
\end{cases}
\]

(1)
Lattice formulation:

- Compact link variables $U_{\mu,x} \in U(1)$, and plaquette gauge action

- **Overlap Hypercube Dirac operator**

  \[
  D_{\text{ovHF}}(m) = \left( 1 - \frac{m}{2} \right) D_{\text{ovHF}}^{(0)}
  \]

  \[
  D_{\text{ovHF}}^{(0)} = 1 + \frac{(D_{\text{HF}} - 1)}{\sqrt{(D_{\text{HF}}^{\dagger} - 1)(D_{\text{HF}} - 1)}}
  \]

Hypercube fermion operator $D_{\text{HF}}(U)$: truncated perfect couplings to nearest neighbours plus plaquette diagonals (average over shortest lattice paths), approximately chiral \[3\].

Insert $D_{\text{HF}}$ into overlap formula \[4\]

$\rightarrow D_{\text{ovHF}}^{(0)}$ obeys the (simplest) Ginsparg-Wilson relation.
The overlap-HF has been applied in quenched QCD [5]. The HF was also used dynamically in finite temperature QCD [6].

In the 2-flavour Schwinger model $D_{ovHF}$ has been simulated with quenched re-weighted configurations and also HMC tests appeared [7, 8, 9].

Comparison with standard overlap operator [4]: computational overhead in the kernel ($D_{HF}$ instead of $D_{Wilson}$), but $D_{ovHF}$ has the following virtues:

- Faster convergence in the polynomial evaluation of $D_{ovHF}$ [3, 7, 5].
- Higher degree of locality and approximate rotation sym. [3, 7, 5, 9].
- Better scaling behaviour [7].

These virtues are all based on the similarity

$$D_{ovHF} \approx D_{HF}.$$ 

(2)

Moreover that property also facilitates HMC simulations:
A preconditioned HMC algorithm

Our concept follows the simplified HMC force for improved staggered fermions of the HF-type [10].

The fermionic force of the standard HMC algorithm

\[
\bar{\Psi} Q_{\text{ovHF}}^{-1} \left[ Q_{\text{ovHF}}^{-1} \frac{\partial Q_{\text{ovHF}}}{\partial A_{x,\mu}} + \frac{\partial Q_{\text{ovHF}}}{\partial A_{x,\mu}} Q_{\text{ovHF}}^{-1} \right] Q_{\text{ovHF}}^{-1} \Psi,
\]

with the Hermitian operator \( Q_{\text{ovHF}} = \gamma_5 D_{\text{ovHF}} \), is simplified to

\[
\bar{\Psi} Q_{\text{ovHF},\varepsilon}^{-1} \left[ Q_{\text{ovHF},\varepsilon}^{-1} \frac{\partial Q_{\text{HF}}}{\partial A_{x,\mu}} + \frac{\partial Q_{\text{HF}}}{\partial A_{x,\mu}} Q_{\text{ovHF},\varepsilon}^{-1} \right] Q_{\text{ovHF},\varepsilon}^{-1} \Psi.
\]

\( Q_{\text{ovHF},\varepsilon} \) approximates \( Q_{\text{ovHF}} \) to precision \( \varepsilon = 10^{-5} \).

This approximation is **useful and cheap** thanks to relation (2).
• The Metropolis accept/reject step uses $Q_{ovHF}$ to machine precision ($10^{-16}$), which renders the algorithm exact.

• Our experience at $\beta = 5$ on a $16 \times 16$ lattice, with trajectory length $\tau = 1/8 = 20 \cdot \Delta \tau$, was reported in Ref. [9].

• Applying the Sexton-Weingarten integration scheme [11], we found acceptance rates $0.3 \ldots 0.5$ for masses $m = 0.01 \ldots 0.24$.

• Reversibility holds to a good precision.
We also confirm **excellent locality**, which is stable in $m$:

\[ f(r) : \text{max. impact of a unit source over distance } r, \langle f(r) \rangle \text{ at } \beta = 5. \]
Statistics:

In view of an $\epsilon$-regime interpretation, we collected large statistics for 7 masses in the sectors with top. charge $\nu = 0$ and $|\nu| = 1$ (index of $D_{ovHF}$).

| $m$  | $\nu = 0$ | $|\nu| = 1$ | total  | topological transitions |
|------|-----------|-----------|-------|------------------------|
| 0.01 | 2079      | 584       | 2663  | 3                      |
| 0.03 | 1105      | 563       | 1668  | 2                      |
| 0.06 | 752       | 711       | 1398  | 5                      |
| 0.09 | 957       | 546       | 1504  | 7                      |
| 0.12 | 699       | 532       | 1505  | 8                      |
| 0.18 | 830       | 609       | 1493  | 13                     |
| 0.24 | 639       | 1030      | 1757  | 17                     |
In the $\epsilon$-regime, chiral Random Matrix Theory [12] relates $\Sigma$ to the ratio of the leading non-zero Dirac eigenvalues in the top. sectors $0$ and $1$:
Combination of this RMT relation with $\Sigma(m)$ in eq. (1)

$\Rightarrow$ prediction for $\frac{\langle \lambda_{|\nu|=1} \rangle}{\langle \lambda_{\nu=0} \rangle}(m)$ vs. numerical data

$m > 0.15$ : approach the $p$-regime behaviour (insensitive to $\nu$) and $m \ll \beta^{-1/2}$ fails.

At $m \leq 0.12$ the data match the predictions well.
Conclusions

• The overlap hypercube fermion has some computational overhead compared to the standard overlap fermion, but a number of benefits: better locality, approximate rotation symmetry, improved scaling, cheap evaluation of the overlap operator and applicability of a simplified HMC force.

• On a $16 \times 16$ lattice at $\beta = 5$ we obtain useful acceptance rates and reversibility.

• To study the $\epsilon$-regime of the Schwinger model, we cumulated statistics at masses $m = 0.01, \ldots, 0.24$ in the sectors of top. charge $|\nu| = 0$ and 1.

• We combined
  - RMT predictions in the $\epsilon$-regime for $\Sigma$ as a function of $\langle \lambda_{|\nu|=1} \rangle / \langle \lambda_{\nu=0} \rangle$
  - Low energy formulae for $\Sigma(m)$
  to predict the mass dependence of the Dirac eigenvalue ratio.

• At $m \leq 0.12$ these predictions match the data remarkably well, although they involve ingredients from the $\epsilon$-regime and from infinite volume.
• Appendix: The HMC algorithm is also implemented for $D_{\text{ovHF}}$ in QCD, with the full HF force. We display thermalisation histories of the dynamical overlap-HF in QCD, on $8^4$ lattices with Zolotarev polynomials of degrees $p = 6$ and $8$ applied in the spectral interval $[lb, \lambda_{\text{max}}]$, $\varepsilon = 10^{-4}$ (cf. p.5), traj. lenght = 1.

![Action histories at L = T = 8, $\beta = 5.6$](image)

- $m=0.1$, $lb=0.1$, $p = 6$, cold start
- $m=0.1$, $lb=0.1$, $p = 6$, hot start
- $m=0.05$, $lb=0.05$, $p = 8$, cold start
- $m=0.05$, $lb=0.05$, $p = 8$, hot start
At $\beta = 5.6$ thermalisation sets in without problems, whereas $\beta = 5.7$ and $\beta = 5.8$ are plagued by a first order phase transition.

We plan applications in the $\epsilon$-regime of QCD, along the lines of Refs. [13].
References