A strategy for performing non-pert. computations in HQET with dynamical light quarks

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Outline

1. Non-perturbative Heavy Quark Effective Theory
2. Implementation in a concrete example: The bottom quark mass
3. Sketch & Status of the computation in two-flavour QCD
Non-perturbative Heavy Quark Effective Theory
Non-perturbative formulation of HQET

Beyond the static approximation

\[ S_{\text{HQET}} = a^4 \sum_x \left\{ \mathcal{L}_{\text{stat}}(x) + \sum_{\nu=1}^{n} \mathcal{L}^{(\nu)}(x) \right\} , \quad \mathcal{L}^{(\nu)}(x) = \sum_i \omega_i^{(\nu)} \mathcal{L}_i^{(\nu)}(x) \]

\[ \mathcal{L}_{\text{stat}} = \bar{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]

\[ \mathcal{L}_1^{(1)} = \bar{\psi}_h \left( -\frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \{ \text{chromomagnetic interaction with the gluon field} \} \]

\[ \mathcal{L}_2^{(1)} = \bar{\psi}_h \left( -\frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \{ \text{kinetic energy from heavy quark’s residual motion} \} \]
Non-perturbative formulation of HQET

Beyond the static approximation

\[ S_{\text{HQET}} = a^4 \sum_{x} \left\{ \mathcal{L}_{\text{stat}}(x) + \sum_{\nu=1}^{n} \mathcal{L}^{(\nu)}(x) \right\}, \quad \mathcal{L}^{(\nu)}(x) = \sum_{i} \omega_{i}^{(\nu)} \mathcal{L}_{i}^{(\nu)}(x) \]

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- \( \delta m, \omega_{i}(g_{0}, m) \) must be determined such that HQET matches QCD
  
  [At the classical level: \( \omega_{\text{spin}} = \omega_{\text{kin}} = 1/m + O(g_{0}^2) \), \( \delta m = 0 + O(g_{0}^2) \)]

- Analogously: Composite fields in the effective theory, e.g.

\[ A_{0}^{\text{HQET}}(x) = Z_{A}^{\text{HQET}} \bar{\psi}_{l}(x) \gamma_{0} \gamma_{5} \psi_{h}(x) + c_{A}^{\text{HQET}} \bar{\psi}_{l}(x) \gamma_{j} \gamma_{5} \hat{D}_{j} \psi_{h}(x) + \ldots \]

\[ \propto 1/m \]
**Expectation values**

Path integral representation at the quantum level

\[
\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\varphi] \, O[\varphi] \, e^{-(S_{\text{rel}} + S_{\text{HQET}})} \quad Z = \int \mathcal{D}[\varphi] \, e^{-(S_{\text{rel}} + S_{\text{HQET}})}
\]

Now the *integrand* is expanded in a *power series* in \(1/m\)

\[
\exp\{-S_{\text{HQET}}\} = \\
\exp\{-a^4 \sum_x \mathcal{L}_{\text{stat}}(x)\} \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2} \left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots \right\}
\]

\[
\Rightarrow \quad \langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\varphi] \, e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)} \, O \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\}
\]
### Expectation values

Path integral representation at the quantum level

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] O[\varphi] e^{-(S_{\text{rel}} + S_{\text{HQET}})} \\
\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-(S_{\text{rel}} + S_{\text{HQET}})}
\]

Now the integrand is expanded in a power series in \(1/m\)

\[
\exp\{-S_{\text{HQET}}\} = \\
\exp\{\sum_x L_{\text{stat}}(x)\} \left\{ 1 - a^4 \sum_x L^{(1)}(x) + \frac{1}{2} \left[a^4 \sum_x L^{(1)}(x)\right]^2 - a^4 \sum_x L^{(2)}(x) + \ldots \right\}
\]

\[
\Rightarrow \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] e^{-S_{\text{rel}} - a^4 \sum_x L_{\text{stat}}(x)} O \left\{ 1 - a^4 \sum_x L^{(1)}(x) + \ldots \right\}
\]

### Important (but not automatic) implications of this definition of HQET

- \(1/m\) – terms appear only as \textit{insertions} of local operators
  - \Rightarrow Power counting: \textit{Renormalizability} at any given order in \(1/m\)
- \(\Leftrightarrow\) Existence of the \textit{continuum limit} with \textit{universality}
- Effective theory \(=\) \textit{Continuum asymptotic} expansion in \(1/m\)
**Expectation values**

Path integral representation at the quantum level

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \, O[\varphi] \, e^{-(S_{\text{rel}}+S_{\text{HQET}})}
\]
\[
\mathcal{Z} = \int \mathcal{D}[\varphi] \, e^{-(S_{\text{rel}}+S_{\text{HQET}})}
\]

Now the *integrand* is expanded in a *power series* in \(1/m\)

\[
\exp\{\!-S_{\text{HQET}}\!\} = \exp\left\{\!-a^4 \sum_x L_{\text{stat}}(x)\!\right\} \left\{1 - a^4 \sum_x L^{(1)}(x) + \frac{1}{2} a^4 \sum_x L^{(1)}(x)^2 - a^4 \sum_x L^{(2)}(x) + \ldots\right\}
\]

\[
\Rightarrow \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \, e^{-S_{\text{rel}}-a^4 \sum_x L_{\text{stat}}(x)} \, O \left\{1 - a^4 \sum_x L^{(1)}(x) + \ldots\right\}
\]

Explicitly:

\[
\langle O \rangle = \langle O \rangle_{\text{stat}} + \omega_{\text{kin}} a^4 \sum_x \langle O O_{\text{kin}}(x) \rangle_{\text{stat}} + \omega_{\text{spin}} a^4 \sum_x \langle O O_{\text{spin}}(x) \rangle_{\text{stat}}
\]

\[
\equiv \langle O \rangle_{\text{stat}} + \omega_{\text{kin}} \langle O \rangle_{\text{kin}} + \omega_{\text{spin}} \langle O \rangle_{\text{spin}}
\]

\[
\langle O \rangle_{\text{stat}} = \frac{1}{\mathcal{Z}} \int_{\text{fields}} \, \mathcal{O} \exp\left\{\!-a^4 \sum_x \left[ L_{\text{light}}(x) + L_{\text{h}}^{\text{stat}}(x) \right]\!\right\}
\]
Mass renormalization pattern in HQET

Already at the level of \( \mathcal{L}_{\text{stat}}(x) = \bar{\psi}_h(x) \left[ \nabla_0^* + \delta m \right] \psi_h(x) \):

Linear divergence \( \delta m \propto a^{-1} \) originates from mixing of \( \bar{\psi}_h D_0 \psi_h \) with \( \bar{\psi}_h \psi_h \)

\[
m_b^{\overline{\text{MS}}} = Z^{\overline{\text{MS}}} \{ m_{\text{bare}} + \delta m \} \quad m_{\text{bare}} = m_B - E_{\text{stat}}
\]

\[
\begin{bmatrix}
m_B : \text{(exp.) B-meson mass} \\
E_{\text{stat}} : \text{static binding energy}
\end{bmatrix}
\]

\[
\delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \left\{ c_1 g_0^2 + c_2 g_0^4 + \ldots \right\}
\]
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\[
m_{\text{bare}}^{\overline{\text{MS}}} = Z^{\overline{\text{MS}}} \{ m_{\text{bare}} + \delta m \} \quad m_{\text{bare}} = m_B - E_{\text{stat}}
\]

- \( m_B \): (exp.) B-meson mass
- \( E_{\text{stat}} \): static binding energy

\[
\delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0g_0^2)} \times \left\{ c_1g_0^2 + c_2g_0^4 + \ldots \right\}
\]

In PT:
uncertainty = truncation error \( \sim e^{1/(2b_0g_0^2)} c_{n+1}g_0^{2n+2} \xrightarrow{g_0 \to 0} \infty \)!

\( \Rightarrow \) **NP renormalization (resp. matching to QCD) of HQET required for the continuum limit to exist**

- Power-law divergences even worse at the level of \( 1/m \) - corrections:
  \( a^{-1} \to a^{-2} \)
We want: \( Lm_b \gg 1 \) to apply HQET & \( \alpha m_b \ll 1 \) to keep \( \alpha \)-effects small
We want: \( Lm_b \gg 1 \) to apply HQET & \( am_b \ll 1 \) to keep \( a- \) effects small

Objection: Need to treat/simulate the b-quark as particle with finite mass
We want: \( L m_b \gg 1 \) to apply HQET & \( a m_b \ll 1 \) to keep \( a \)-effects small

Objection: Need to treat/simulate the \( b \)-quark as particle with finite mass

⇒ Trick: Start with QCD in a small volume \( V = L^4 \), \( L \equiv L_1 \simeq 0.4 \text{ fm} \)

Matching conditions

\[
\Phi^\text{QCD}_k = \Phi^\text{HQET}_k
\]

for observables \( \Phi_k \)

(renormalized quantities, computable for \( a \to 0 \))
NP HQET

NP matching in *finite* volume

[ H. & Sommer, JHEP0402(2004)022 ]

**QCD**

\[ \frac{1}{m_b} \gg a \]

**HQET**

\[ \frac{1}{m_b} \ll L \]

Matching conditions

\[ \Phi_{QCD}^k = \Phi_{HQET}^k \]

for observables \( \Phi_k \)

(renormalized quantities, computable for \( a \to 0 \))

- HQET parameters fixed by relating them to QCD observables in small \( V \)
- Sound approach, because
  - the underlying Lagrangian does not know about the finite \( V \)!
  - rather than having a propagating ‘real’ relativistic b-quark, one aims at determining the *NP heavy quark mass dependence* of \( \Phi_{QCD}^k \)
Connecting small and large volumes

- Matching volume: \( L = L_1 \approx 0.4 \text{ fm} \), very small lattice spacings
- Gap to large volumes and practicable lattice spacings, where physical quantities (e.g. \( m_B \), \( F_{B_s} \)) may be extracted, bridged by a . . .
Connecting small and large volumes

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Finite-size scaling step

[ Lüscher, Weisz & Wolff, 1991; \( \text{Alpha Collaboration} \), 1993-2006 ]

HQET in large volume:

\[
\begin{align*}
&\Phi_{k}^\text{HQET} (L_2, M_b) \\
&\Phi_{k}^\text{QCD} (L_1, M_b)
\end{align*}
\]

Matching

\[ \sigma_k \]

0.4 fm
Finite-size scaling step

HQET in large volume:

$m_B, f_B, \ldots$

$\Phi_{\text{HQET}}^k(L_2, M_b)$

$0.4 \text{ fm}$

$\Phi_{\text{QCD}}^k(L_1, M_b)$

matching

$\sigma_k$

Use the \textit{QCD Schrödinger Functional}, $L \rightarrow 2L$ via Step Scaling Functions

$\Phi_{\text{HQET}}^k(2L) = \sigma_k \left( \{ \Phi_{\text{HQET}}^j(L), j = 1, \ldots, N \} \right)$

$2L = 2L_1 \simeq 0.8 \text{ fm}$

$\rightarrow$ Large $V$ ($L \simeq 2 \text{ fm}$) at same resolution, where a $B$-meson fits comfortably

- Fully non-perturbative, continuum limit can be taken everywhere
Implementation in a concrete example:
The bottom quark mass
Computation of $M_b$

Non-trivial matching problem:

\[
\{ m_{\text{bare}} + \delta m, \omega_{\text{kin}}, \omega_{\text{spin}} \} \quad \longleftrightarrow \quad M_b
\]

$\Rightarrow N = 3$ matching conditions:

\[
\Phi^{\text{QCD}}_k (L, M) = \Phi^{\text{HQET}}_k (L, M) \quad k = 1, 2, 3
\]

[$M$ : RGI heavy quark mass]
Non-trivial matching problem:

\[ \{ m_{\text{bare}} + \delta m, \omega_{\text{kin}}, \omega_{\text{spin}} \} \quad \leftrightarrow \quad M_b \]

\[ \Rightarrow N = 3 \text{ matching conditions:} \]

\[ \Phi_{k}^{\text{QCD}}(L, M) = \Phi_{k}^{\text{HQET}}(L, M) \quad k = 1, 2, 3 \]

\[ [M : \text{RGI heavy quark mass}] \]

**Basic equation in leading order of HQET (static approximation)**

\[ m_B = E_{\text{stat}} - E_{\text{stat}} + E_{\text{stat}} \quad \text{with} \quad E_{\text{stat}} = E(L_1, M_b) \quad \text{[matching to QCD]} \]

\[ = E_{\text{stat}} - E_{\text{stat}}(L_2) + E_{\text{stat}}(L_2) - E_{\text{stat}}(L_1) + E(L_1, M_b) \quad \text{(*)} \]

\[ \alpha \to 0 \text{ in HQET} \quad \alpha \to 0 \text{ in HQET (}\sigma_m\text{)} \quad \alpha \to 0 \text{ } \Phi_2/L_1 \text{ in QCD} \]

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
- \( \Phi_2(L_1, M) \) carries entire (relativistic) heavy quark mass dependence
Basic equation in leading order of HQET (static approximation)

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with \( E_{\text{sub stat}} = E(L_1, M_b) \)

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\( \alpha \to 0 \) in HQET \( \alpha \to 0 \) in HQET \( (\sigma_m) \)

\( \Phi_2/L_1 \) in QCD

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Quenched result

Della Morte, Garron, Sommer & Papinutto
JHEP01(2007)007

- \( L_1 \simeq 0.4 \text{ fm, } L_2 = 2L_1 \)
- Use \( r_0 m_B^{(\text{exp})} \), \( r_0 = 0.5 \text{ fm} \) & solve (\( \star \))

\[ \Rightarrow \ M_b^{\text{stat}} = (6771 \pm 99) \text{ MeV} \]

- NP renormalization & Continuum limit
- Error dominated by that on \( Z_M \) (\( \simeq 1\% \)) in \( LM = Z_M Z (1 + b_m am_q) \times Lm_q \)
Basic equation in leading order of HQET (static approximation)

\[ m_B = E_{\text{stat}} - E_{\text{stat}}^{\text{sub}} + E_{\text{sub}}^{\text{stat}} \quad \text{with} \quad E_{\text{stat}}^{\text{sub}} = E(L_1, M_b) \quad \text{[matching to QCD]} \]

\[ = E_{\text{stat}} - E_{\text{stat}}(L_2) + E_{\text{stat}}(L_2) - E_{\text{stat}}(L_1) + E(L_1, M_b) \quad (\star) \]

\( \alpha \to 0 \) in HQET \quad \alpha \to 0 \) in HQET \quad (\sigma_m) \quad \alpha \to 0 \equiv \Phi_2/L_1 \) in QCD

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
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  - NP renormalization & Continuum limit
  - Error dominated by that on \( Z_M \) \( (\simeq 1\%) \) in
  \[ LM = Z_M Z (1 + b_m a m_q) \times L m_q \]
Inclusion of $1/m$–terms

Della Morte, Garron, Sommer & Papinutto, JHEP0701(2007)007

$m_B$ at next-to-leading order of HQET

\[ m_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}} \]

- $E_{\text{kin}}, E_{\text{spin}}$ associated with $\bar{\psi}_h (-\frac{1}{2} D^2) \psi_h$ and $\bar{\psi}_h (-\frac{1}{2} \sigma \cdot B) \psi_h$ in $\mathcal{L}^{(1)}$
  \[ \rightarrow \text{Three observables } \Phi_1, \Phi_2, \Phi_3 \text{ required in the matching step} \]
- Considering the spin-averaged $B$-meson instead, $\omega_{\text{spin}}$ cancels:
  \[ m_B^{(\text{av})} = \frac{1}{4} m_B + \frac{3}{4} m_B^* = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} \]
  \[ \rightarrow \text{Only two observables } \Phi_1, \Phi_2 \text{ necessary} \]
Inclusion of 1/m-terms

Della Morte, Garron, Sommer & Papinutto, JHEP0701(2007)007

\( m_B \) at next-to-leading order of HQET

\[
m_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}}
\]

- \( E_{\text{kin}}, E_{\text{spin}} \) associated with \( \bar{\psi}_h(-\frac{1}{2}D^2)\psi_h \) and \( \bar{\psi}_h(-\frac{1}{2}\sigma \cdot B)\psi_h \) in \( \mathcal{L}^{(1)} \)

→ Three observables \( \Phi_1, \Phi_2, \Phi_3 \) required in the matching step

- Considering the spin-averaged B-meson instead, \( \omega_{\text{spin}} \) cancels:

\[
m_{B}^{(\text{av})} = \frac{1}{4} m_B + \frac{3}{4} m^* = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}}
\]

→ Only two observables \( \Phi_1, \Phi_2 \) necessary

### Strategy

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Lattice with ( am_b \ll 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_B = 5.4 ) GeV</td>
<td>( L_1 \simeq 0.4 ) fm, ( L_2 = 2L_1 )</td>
</tr>
<tr>
<td>( \Phi_1^{\text{HQET}}(L_2), \Phi_2^{\text{HQET}}(L_2) )</td>
<td>( \Phi_1^{\text{HQET}}(L_1), \Phi_2^{\text{HQET}}(L_1) )</td>
</tr>
<tr>
<td>( u_1 = \bar{g}^2(L_1) )</td>
<td>( \sigma_m(u_1), \sigma_{\text{kin}}^1(u_1), \sigma_{\text{kin}}^2(u_1) )</td>
</tr>
<tr>
<td>( \sigma_m(u_1) )</td>
<td>( \Phi_1(L_1, M), \Phi_2(L_1, M) )</td>
</tr>
</tbody>
</table>
Matching formula = Static part + $1/m$–correction:

$$L_2 m_B^{(av)} = L_2 m_B^{stat} + L_2 [E_{stat} - \Gamma_{stat}^1(L_2)] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b)$$

$$+ L_2 m_B^{(1)} + L_2 [E_{kin} - \Gamma_{kin}^1(L_2)] \omega_{kin}$$
Matching formula = Static part + $1/m$ - correction:

$$L_2 m_B^{(\text{av})} = L_2 m_B^{\text{stat}} \right\} = L_2 \left[ E_{\text{stat}} - \Gamma_{1}^{\text{stat}}(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M_B)$$

$$+ L_2 m_B^{(1)} \right\} = \sigma_2^{\text{kin}}(u_1)\Phi_1(L_1, M_B) + L_2 \left[ E_{\text{kin}} - \Gamma_{1}^{\text{kin}}(L_2) \right] \omega_{\text{kin}}$$

Implementation

- **SF boundary conditions**, i.e. $T \times L^3$, Dirichlet at $x_0 = 0$, $T$, fermion fields periodic in space modulo a phase: $\psi(x + L\hat{k}) = e^{i\theta}\psi(x)$

- Employ (finite-volume) B-meson energies for the matching:
  Avoid $c_A^{\text{HQET}} f_{\text{stat}}^{A} \delta_A$ - term in boundary-to-$\Lambda_0$ correlator by boundary-to-boundary ones with pseudoscalar or vector quantum numbers
Matching formula \(=\) Static part \(+\) \(1/m\) correction:

\[
\begin{align*}
L_2 m_B^{(αν)} &= L_2 m_B^{\text{stat}} \quad \left\{ \begin{array}{l}
= L_2 \left[ E_{\text{stat}} - \Gamma_1^{\text{stat}}(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b) \\
+ L_2 m_B^{(1)} \quad \left\{ \begin{array}{l}
= \sigma_2^{\text{kin}}(u_1)\Phi_1(L_1, M_b) + L_2 \left[ E_{\text{kin}} - \Gamma_1^{\text{kin}}(L_2) \right] \omega_{\text{kin}}
\end{array} \right.
\end{array} \right.
\end{align*}
\]

Implementation

- **SF boundary conditions**, i.e. \(T \times L^3\), Dirichlet at \(x_0 = 0\), \(T\), fermion fields periodic in space modulo a phase: \(ψ(x + L\hat{k}) = e^{iθ}ψ(x)\)

- Employ (finite-volume) B-meson energies for the matching:

Avoid \(c_A^{\text{HQET}} f_\delta^{\text{stat}}\) term in boundary-to-\(A_0\) correlator by boundary-to-boundary ones with pseudoscalar or vector quantum numbers

\[
(f_A)_R (x_0) = Z_A^{\text{HQET}} Z_{\zeta h} Z_{\zeta} e^{-m_{\text{bare}}x_0} \left\{ f_A^{\text{stat}} + \omega_{\text{kin}} f_A^{\text{kin}} + \omega_{\text{spin}} f_A^{\text{spin}} + c_A^{\text{HQET}} f_A^{\text{stat}} \right\}
\]
Matching formula = Static part + $1/m$ correction:

$$L_2 m_B^{(av)} = \begin{cases} L_2 m_B^{stat} \\ + L_2 m_B^{(1)} \end{cases} = L_2 \left[ E_{stat} - \Gamma_1^{stat}(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b)$$

Implementation

- SF boundary conditions, i.e. $T \times L^3$, Dirichlet at $x_0 = 0$, $T$, fermion fields periodic in space modulo a phase: $\psi(x + L\hat{k}) = e^{i\theta} \psi(x)$
- Employ (finite-volume) B-meson energies for the matching:
  Avoid $c_A^{HQET} f_1^{stat} - \text{term in boundary-to-}A_0 \text{ correlator by boundary-to-boundary ones with pseudoscalar or vector quantum numbers}$

$$ (f_1)_R(T) = Z_{\zeta_h}^2 Z_\zeta^2 e^{-m_{bare}\Gamma_1} \left\{ f_1^{stat} + \omega_{kin} f_1^{kin} + \omega_{spin} f_1^{spin} \right\}$$
Matching formula = Static part + $1/m$–correction:

$$L_2 m_{B}^{(av)} = \{ L_2 m_B^{\text{stat}} \} = \left[ E_{\text{stat}} - \Gamma_{1}^{\text{stat}}(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b)$$

$$+ \left[ E_{\text{kin}} - \Gamma_{1}^{\text{kin}}(L_2) \right] \omega_{\text{kin}}$$

**Implementation**

- **SF boundary conditions**, i.e. $T \times L^3$, Dirichlet at $x_0 = 0$, $T$, fermion fields periodic in space modulo a phase: $\psi(\chi + L\hat{k}) = e^{i\theta}\psi(\chi)$

- **Employ (finite-volume) B-meson energies for the matching:** Avoid $\frac{\delta A}{A} f_{\text{stat}}^\text{stat}$ – term in boundary-to-$A_0$ correlator by boundary-to-boundary ones with pseudoscalar or vector quantum numbers

$$\left( f_1 \right)_R(T) = Z_h^2 Z_\zeta^2 e^{-m_{\text{bare}}T} \left\{ f_1^{\text{stat}} + \omega_{\text{kin}} f_1^{\text{kin}} + \omega_{\text{spin}} f_1^{\text{spin}} \right\}$$

- **$\Phi_1$**: Suitable ratios of $f_1(\theta), f_1(\theta')$ such that $Z$–factors drop out

- **$\Phi_2$**: *Spin-averaged* energy $\Gamma_1 = -\partial_T \ln f_1^{(av)} = m_{\text{bare}} + \Gamma_{1}^{\text{stat}} + \omega_{\text{kin}}\Gamma_{1}^{\text{kin}}$
A few more details

Boundary-to-boundary SF correlation functions and energies:

\[ f_1(\theta) = -\frac{a^{12}}{2L^6} \sum_{u,v,y,z} \langle \bar{\zeta}_l'(u)\gamma_5 \bar{\zeta}_b'(v) \bar{\zeta}_b(y)\gamma_5 \zeta_l(z) \rangle \]

\[ k_1(\theta) = -\frac{a^{12}}{6L^6} \sum_{u,v,y,z,k} \langle \bar{\zeta}_l'(u)\gamma_k \bar{\zeta}_b'(v) \bar{\zeta}_b(y)\gamma_k \zeta_l(z) \rangle \]

\[ F_1(L, \theta) = \frac{1}{4} \left[ \ln f_1(\theta) + 3 \ln k_1(\theta) \right] \]

\[ R_1(L, \theta_1, \theta_2) = F_1(L, \theta_1) - F_1(L, \theta_2) \bigg|_{T=L/2} \]

\[ \Gamma_1(L, \theta_0) = -\frac{\partial T + \partial^* T}{2} F_1(L, \theta_0) \bigg|_{T=L/2} \]
A few more details

From these, form dimensionless observables

\[ \Phi_1(L, M_b) = R_1(L, \theta_1, \theta_2) - R_{1\text{stat}}(L, \theta_1, \theta_2) \]
\[ \Phi_2(L, M_b) = L \Gamma_1(L, \theta_0) \]
\[ R_{1\text{stat}}(L, \theta_1, \theta_2) = \ln \left[ \frac{f_{1\text{stat}}(L, \theta_1)}{f_{1\text{stat}}(L, \theta_2)} \right] \bigg|_{T=L/2} \]

that have $1m_b$ – expansions

\[ \Phi_1(L, M_b) = \omega_{\text{kin}} R_{1\text{kin}}(L, \theta_1, \theta_2) \]
\[ \Phi_2(L, M_b) = L \left[ m_{\text{bare}} + \Gamma_{1\text{stat}}(L, \theta_0) + \omega_{\text{kin}} \Gamma_{1\text{kin}}(L, \theta_0) \right] \]

Moreover

- define SSFs $\sigma_m(u), \sigma_{1\text{kin}}(u), \sigma_{2\text{kin}}(u)$ in the effective theory, and
- fit large $-x_0$ asymptotics of the large-volume B-meson energy to

\[ \Gamma(x_0) = E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + (A_{\text{stat}} + \omega_{\text{kin}} A_{\text{kin}}) e^{-\Delta_{\text{stat}} x_0} (1 - \omega_{\text{kin}} x_0 \Delta_{\text{kin}}) + \ldots \]
**A few more details**

**Most difficult piece encountered in the quenched calculation:**

Large-volume HQET matrix element \( [E_{\text{kin}} - \Gamma_1^{\text{kin}}(L_2)] \) entering one of the \( 1/m_b \) contributions.
A few more details

Most difficult piece encountered in the quenched calculation:
Large-volume HQET matrix element \( [E_{\text{kin}} - \Gamma_{1}^{\text{kin}}(L_{2})] \) entering one of the \( 1/m_{b} \) – contributions

Result in the quenched approximation

\[
\overline{m}_{b}^{\overline{MS}}(\overline{m}_{b}) = 4.374(64) \text{ GeV} - 0.027(22) \text{ GeV} + \frac{O(\Lambda^{3}/m_{b}^{2})}{O(\Lambda^{2}/m_{b})} + \text{negligible}
\]
Sketch & Status of the computation in two-flavour QCD
Physics goals

- $m_b^{(N_f=2)}$
- $F_{B_s}^{(N_f=2)}$

from lattice HQET including $1/m_b$ – corrections

General setup of the dynamical finite-volume simulations:

- QCD with Schrödinger Functional boundary conditions ($T$, $L$, $\theta$)
- $N_f = 2$ degenerate massless sea quarks ($m_l \equiv m_{\text{light}} = 0$)
- Evaluation of heavy-light correlation functions where the heavy (valence) quark is quenched
- Configurations generated on apeNEXT @ DESY-Zeuthen
Elements of the computation

(1) Matching to QCD with a relativistic b-quark

- Choice of the matching volume resp. \( L_1 \):
  - \( L_1 \) such that \( L_1/r_0 \approx 1 \), i.e. \( L_1 \approx (0.4 - 0.5) \text{ fm} \)
  - \( \Rightarrow \) connection to large volume possible after one step scaling step

\[
L_1 \xrightarrow{\sigma_k} L_2 = 2L_1 \quad L_\infty = 4L_1 \approx 2 \text{ fm}
\]

- Based on the knowledge of the \( N_f = 2 \) running of the renormalized SF coupling \( \bar{g}^2 \equiv g_{SF}^2 \) (resp. its SSF \( \sigma \)) from \([\text{\textsc{alpha}} \text{ Collaboration}, \text{NPB713(2005)378}]\)

\( L_1 \) is fixed by the condition

\[
\bar{g}^2(L_0) = \text{constant} \approx 3.0 \quad \text{where} \quad L_0 = L_1/2
\]

such that

\[
u_1 = \bar{g}^2(L_1) = \sigma(\bar{g}^2(L_0)) \approx 4.3
\]

\( \Rightarrow (L_1/(2a), \beta, \kappa_1) \) with \( L_1/(2a) = 10, 12, 16, 20 \) & \( m_1^{\text{PCAC}}(L_1/2) = 0 \)
Elements of the computation

(1) Matching to QCD with a relativistic b-quark

- NP calculation of the heavy quark mass dependence of heavy-light meson observables in (the cont. limit of) finite-volume lattice QCD:
  - $L_1/a = 20, 24, 32, 40$, $T = L_1$, same $\beta$'s and $m^{\text{PCAC}}_f(L_1) = 0$
  - Fix the RGI heavy quark mass to values around the b-quark via

\[
z \equiv L_1 M = L_1 \times Z_m \times \frac{M}{m(\mu_0)} \times m_q (1 + b_m a m_q), \quad \mu_0 = \frac{2}{L_1}
\]

\[\text{[ ALPHA Collaboration, NPB729(2005)117]}\]

- $a m_q = \frac{1}{2} \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right)$

\[Z_m = \frac{Z(g_0) Z_A(g_0)}{Z_P(g_0, L_1/(2a))}\]

⇒ demands to determine $b_m(g_0)$ and $Z(g_0)$

- In progress . . .
For $1/m$ corrections:

- $-\partial_T \ln f_1^{(\alpha \nu)}$ requires additional lattices with $T = L_1/2$, $T = L_1/2 \pm a$
- however, these simulations will be by a factor $\simeq 4$ less expensive
For $1/m$ corrections:

- $-\partial_T \ln f_1^{(\alpha)}$ requires additional lattices with $T = L_1/2$, $T = L_1/2 \pm \alpha$
- However, these simulations will be by a factor $\sim 4$ less expensive

For the HQET side, remember the general strategy . . .
Elements of the computation — 2-Do’s & Prospects

- For $1/m$-corrections:
  - $-\partial_T \ln f_1^{(av)}$ requires additional lattices with $T = L_1/2$, $T = L_1/2 \pm a$
  - however, these simulations will be by a factor $\simeq 4$ less expensive

(2) Connection to $L_2 = 2L_1$ in HQET

- Through SSFs $\sigma_m(u_1), \sigma_1^{\text{kin}}(u_1), \sigma_2^{\text{kin}}(u_1)$, after some tuning:
  - $(L_1/a, \beta), (L_2/a, \beta)$ s. th. $\bar{g}^2(L_1), \bar{g}^2(L_2)$ fixed ($L_1/a = (6), 8, 10, 12$)

- Outcome for $5.2 \lesssim \beta \lesssim 5.6$:
  - $\delta m, \omega_{\text{kin}}, \omega_{\text{spin}}$
  - $Z_A^{\text{HQET}}, c_A^{\text{HQET}}$
Elements of the computation — 2-Do’s & Prospects

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  - $Z_A^{\text{HQET}}$, $c_A^{\text{HQET}}$

(3) HQET observables in physically large volume, $L_\infty = 4L_1 \approx 2$ fm

- $E_{\text{stat}}$, $E_{\text{kin}}$, $E_{\text{spin}}$, . . . :
  - periodic boundary conditions, all-to-all propagators à la Dublin
  - generate, share & use existing configurations