Excited Nucleon Resonances from the Lattice

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Lattice Hadron Physics Collaboration

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Goal: To use the Monte Carlo method to calculate

- the low-lying spectrum of QCD hadron resonances
- form factors, structure functions, and other matrix elements
- hadron-hadron interactions
Two-point functions provide access the spectrum of the theory

\[ C(\tau) \equiv \langle \mathcal{O}(\tau) \overline{\mathcal{O}}(0) \rangle \equiv \frac{1}{Z} \sum_{n=0}^{\infty} \langle n | e^{-(\beta-\tau)H} \mathcal{O} e^{-\tau H} \overline{\mathcal{O}} | n \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-(\beta-\tau)E_n} \langle n | \mathcal{O} e^{-\tau H} \overline{\mathcal{O}} | n \rangle \]

For this work, take \( \beta \gg \tau \) to access zero-temperature physics

\[ C(\tau) \overset{\beta \gg \tau}{\to} \langle 0 | \mathcal{O} e^{-\tau H} \overline{\mathcal{O}} | 0 \rangle \]
Insert a complete set of energy states to see the spectral structure

\[ C(\tau) = \langle 0 | \mathcal{O} e^{-H\tau} \mathcal{O} | 0 \rangle \]

\[ = \langle 0 | \mathcal{O} \sum_{k=0}^{\infty} |k\rangle \langle k| e^{-H\tau} \mathcal{O} | 0 \rangle \]

\[ = \sum_{k=1}^{\infty} |\langle k| \mathcal{O} | 0 \rangle|^2 e^{-E_k \tau}, \quad \langle 0| \mathcal{O} | 0 \rangle = 0 \]

The goal of this work: extract the spectrum \( \{E_k\} \)
Overview

- Operator construction
  - Local versus extended operators
  - Quark field and gauge link smearing
  - Continuum spin identification (not discussed)

- Extracting excited resonances
  - The standard variational method
  - Variations on the variational method
  - Toy model

- Nucleon spectrum preliminary results

- Outlook and conclusions
Part I

*Operator construction*
Baryon constituents determine total angular momentum $J$

$$\vec{J} = \vec{S}_q + \vec{S}_g + \vec{L}$$

$J_{\text{max}} = 3/2$ for a localized operator with unexcited glue ($\vec{S}_g = \vec{L} = 0$)

Need displacements to access higher $J$ quantum numbers ($5/2, 7/2, \ldots$)
<table>
<thead>
<tr>
<th>Operator type</th>
<th>Displacement indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-Site</td>
<td>$i = j = k = 0$</td>
</tr>
<tr>
<td>Singly-Displaced</td>
<td>$i = j = 0, k \neq 0$</td>
</tr>
<tr>
<td>Doubly-Displaced-L</td>
<td>$i = 0, j = -k, k \neq 0$</td>
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<td>Doubly-Displaced-I</td>
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<tr>
<td>Triply-Displaced-T</td>
<td>$i = -j,</td>
</tr>
</tbody>
</table>
Quark field and gauge link smearing

- Quark field smearing damps out high frequency couplings
- Gauge link smearing damps out high frequency fluctuations
- Combining them gives good extended quark operators
Part II

Extracting excited resonances
The variational method

- Trial basis of trial vectors \( \{ |\phi_a\rangle \} \) defines a subspace

\[
|\psi\rangle = |\phi_a\rangle v_a \quad \text{(sum over repeated indices implied)}
\]

- Let \( f(\psi) \) be the expectation value of some Hermitian \( F \) w.r.t. \( |\psi\rangle \)

\[
f(\psi) \equiv \frac{\langle \psi | F | \psi \rangle}{\langle \psi | \psi \rangle}
\]

- The first-order variation with respect to the \( v_a \) gives

\[
\delta f(\psi) = 0 \leftrightarrow F_\parallel |\psi\rangle = f(\psi) |\psi\rangle,
\]

where \( F_\parallel \) is the restriction of \( F \) to the subspace spanned by the \( \{ |\phi_a\rangle \} \)
Using the variational method to extract excited states

- Instead of one operator $\overline{O}$, use a \textit{basis} of operators $\{\overline{O}_a\}$ to define a \textit{correlator matrix}

$$C(\tau) \equiv \langle 0 | \overline{O}_a e^{-H\tau} \overline{O}_b | 0 \rangle$$

- We may formally define several variational bases parameterized by a reference time $\tau_0$

$$|O_a(\tau_0)\rangle \equiv e^{-H\tau_0/2} \overline{O}_a |0\rangle, \quad |\Theta(\tau_0)\rangle \equiv |O_a(\tau_0)\rangle v_a$$

- To diagonalize the transfer matrix within the subspace, find the extremum of

$$\frac{\langle \Theta(\tau_0) | e^{-H(\tau-\tau_0)} | \Theta(\tau_0) \rangle}{\langle \Theta(\tau_0) | \Theta(\tau_0) \rangle} = \frac{v^\dagger C(\tau) v}{v^\dagger C(\tau_0) v}$$
Generalized eigenvalue problem

○ We may find the extrema of

\[ \frac{v^\dagger C(\tau)v}{v^\dagger C(\tau_0)v} \]

○ By solving the generalized Hermitian eigenvalue problem:

\[ C(\tau)v = \lambda C(\tau_0)v, \]

or

\[ C^{-1/2}(\tau_0)C(\tau)C^{-1/2}(\tau_0)u = \lambda u, \quad u \equiv C^{1/2}(\tau_0)v. \]

○ The different solutions \( \lambda(\tau) \) are called \textit{principal correlators}, and each is asymptotically dominated by a unique energy level \( E_k \).

○ This requires good estimates of the elements of \( C(\tau_0) \) and \( C(\tau) \).
For fixed $t_0$, the different solutions asymptote in $t$ to orthogonal levels.

Consider a toy theory with three energy states.

Use two operators to form the correlator matrix.

Effective mass results:

Original operators

Principal correlators

A. C. Lichtl (RBRC-BNL)

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Choosing $t_0$

- Physically $\tau_0$ determines how much we ‘relax’ our basis onto the lower-lying states

$$|\mathcal{O}_a(\tau_0)\rangle \equiv e^{-H\tau_0/2}O_a|0\rangle = \sum_k |k\rangle c_k a e^{-E_k\tau_0/2}$$

- At early times, the basis will suffer from excited state contamination
- At late times, the basis will be ambiguous due to a poor signal-to-noise ratio
Choosing $t_0$ and $t$

- **A:** fix $t_0$ and find solutions $\{|\Theta\rangle\}$ for all possible $t$ to define principal correlators (Lüscher-Wolff)
  - **Pro:** Maintains orthogonality at all $t$
  - **Con:** Must stop at $t_{max}$ where noise begins to dominate

- **B:** fix both $t_0$ and $t$ to define fixed-coefficient correlators (ACL)
  - **Pro:** Can extend fit range further
  - **Con:** Lose orthogonality, more suitable to precondition a matrix fit

- **C:** vary both $t_0$ and $t$ and look at estimates of $e^{-H(\tau-t_0)}$
  - **Pro:** Uses much more information from the correlation matrix
  - **Con:** In progress
Part III

Nucleon spectrum preliminary results
Description of calculation

Stage 1) Exploratory study:

- Objective: refine operator design method, demonstrate feasibility of variational method
- Anisotropic lattice: $\frac{a_T}{a_s} = 3.0$
- Lattice spacing $a_s = 0.1$ fm
- Lattice extent: $12^3 \times 48$
- Spatial volume 1.2 fm
- Pion mass: $m_\pi = 700$
- Quenched
Principal versus fixed-coefficient correlators

- Principal correlators (red) break down at early times
- Fixed-coefficient correlators (green) show little instability
- Degenerate levels may be suspect due to lack of orthogonality
- Fits are in blue
Nucleons: Comparison with experiment

Nucleon Mass Spectrum (Exp)
Nucleons: Comparison with experiment

Nucleon Mass Spectrum (Exp)

Nucleon Mass Spectrum (n_f=0)
Part IV

Outlook and conclusions
Outlook

- Direct fits to the correlators (principal or fixed-coefficient) appear suspect due to contamination effects and signal-to-noise ratio issues (the situation is even worse when dynamical quarks are included).

- Should perform a *truncated matrix fit* of the form:

\[
C_{ab}^{(\text{fit})}(\tau) = \sum_{k=1}^{D} c_{ka}^* c_{kb} e^{-E_k \tau}
\]

- Can precondition using the fixed-coefficient solution by rotating \( C(\tau) \).

- Alternatively, attempt to extract information by varying both \( \tau_0 \) and \( \tau \).
Conclusions

- A quantitative look at the excited QCD spectrum is nearly in reach
- This long-term effort requires careful operator design and sophisticated analysis techniques
- Ongoing work:
  - Validate method in two-flavor sector at two pion masses and two volumes
  - High-statistics runs using three dynamical quark flavors
  - Volume studies to identify single versus multi-particle states

Any Questions?