Deriving confinement via RG decimations

T. Tomboulis

Physics and Astronomy Department
UCLA

30July - August 4, 2007 / Lattice-07
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The problem

- 4-dim. $SU(N)$ gauge theory at $T = 0$ known to be single phase for $0 < \beta < \infty$.
- Hard to derive directly: Multiscale problem – passage from short distance perturbative regime ordered regime to long distance non-perturbative confining disordered regime.
- Natural framework: RG block-spinning procedure bridging the desperate regimes.
- Ideally: Construct exact block-spinning scheme converging to the ‘perfect action’ along Wilsonian renormalized trajectory. Then one can compute any observable at different scales.

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Too ambitious: Technically too complicated to carry out so far.
More modest but sufficiently general framework.

Employ approximate, but easily explicitly computable decimation procedures that can provide bounds on judicially chosen quantities.

Such quantities are partition functions (free energies) and/or their ratios.

Consider only such quantities - they can serve as order parameters.

Do not attempt to construct general RG blocking action suitable for any observable.

Use the bounds to constrain the corresponding exact quantities – interpolate between the bounds to fix on exact quantities.
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Some notation

Start with plaquette action at spacing $a$, for example Wilson action:

$$A_p(U) = \frac{\beta}{2} \text{Re tr} U_p$$

Character expansion of the exponential of plaquette action:

$$e^{A_p(U)} = \sum_j d_j F_j(\beta, a) \chi_j(U)$$

SU(2): $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, $d_j = (2j + 1)$.

In terms of normalized coefficients;

$$e^{A_p(U)} = F_0 \left[ 1 + \sum_{j \neq 0} d_j c_j(\beta) \chi_j(U) \right]$$

Partition Function (PF) on lattice $\Lambda$

$$Z_\Lambda(\beta) = \int dU_\Lambda \prod_p \left[ 1 + \sum_{j \neq 0} d_j c_j(\beta) \chi_j(U) \right] \equiv Z_\Lambda(\{c_j(\beta)\})$$
The RG decimation can be summarized as a set of decimation rules for each successive step:

\[ a \rightarrow ba \rightarrow b^2 a \rightarrow \cdots \rightarrow b^n a \]

\[ \Lambda \rightarrow \Lambda^{(1)} \rightarrow \Lambda^{(2)} \rightarrow \cdots \rightarrow \Lambda^{(n)} \]

from lattice \( \Lambda^{(m)} \) of spacing \( b^m a \) to lattice \( \Lambda^{(m+1)} \) of spacing \( b^{m+1} a \).

The rules give explicit expressions for the computation of the Fourier coefficients at the \( m+1 \)-th step given those of the \( m \)-th step:

\[ F_0(m) = F_0(\zeta, r, b, \{c_i(m-1)\}) \]

\[ c_j(m) = c_j(\zeta, r, b, \{c_i(m-1)\}) \]

The rule involves parameters \( \zeta, r \) which control the amount by which the interactions of the remaining plaquettes after a decimation step are ‘renormalized’ to compensate for the ones that were removed.
PF transformation:

$$Z_{\Lambda^{(m-1)}} \left( \{ c_j(m-1) \} \right) \rightarrow F_0(m)^{\Lambda^{(m)}} Z_{\Lambda^{(m)}} \left( \{ c_j(m) \} \right)$$

After each decimation step the resulting action retains the original one-plaquette form but will contain all representations

$$\exp A_p(m) = \left[ 1 + \sum_{j \neq 0} d_j c_j(m) \chi_j(U) \right]$$

with

$$A_p(m) = \sum_j \beta_j(m) \chi_j(U).$$

Also, both positive and negative effective couplings $\beta_j(m)$ will occur. But all $c_j(m) \geq 0$ if $\zeta = \text{integer}$. $\leftrightarrow$ Reflection Positivity
Upper and lower bounds

Go from the $m-1$ step to the $m$ decimation step with decimation parameters:

- (I) $\zeta = b^{d-2}$, $r = 1 - \varepsilon$, $0 \leq \varepsilon < 1$ (MK choice); denote resulting coefficients by $F_0^U(m)$ and $c^U_j(m)$.

- (II) $\zeta = 1$, $r = 1$; denote resulting coefficients by $F_0^L(m) = 1$ and $c^L_j(m)$.

Then

$$F_0^L(m)^{|\Lambda^{(m)}|} Z_{\Lambda^{(m)}} \left( \{ c^L_j(m) \} \right) < Z_{\Lambda^{(m-1)}} < F_0^U(m)^{|\Lambda^{(m)}|} Z_{\Lambda^{(m)}} \left( \{ c^U_j(m) \} \right)$$
Introducing a parameter $\alpha$, $0 \leq \alpha \leq 1$, define interpolating coefficients $\tilde{c}_j(m, \alpha)$ and $\tilde{F}_0(m, \alpha)$ such that

$$\tilde{c}_j(m, \alpha) = \begin{cases} c_j^U(m) : & \alpha = 1 \\ c_j^L(m) : & \alpha = 0 \end{cases}$$

and similarly

$$\tilde{F}_0(m, \alpha) = \begin{cases} F_0^U(m) : & \alpha = 1 \\ F_0^L(m) : & \alpha = 0 \end{cases}$$

But there is nothing unique about any one possible smooth interpolation. Consider more generally a family of smooth interpolations parametrized by a parameter $t$ in some interval $(t_1, t_2)$.
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But there is nothing unique about any one possible smooth interpolation. Consider more generally a family of smooth interpolations parametrized by a parameter $t$ in some interval $(t_1, t_2)$. 
Then the upper-lower bounds statement

\[ F^L_0(m)^{|\Lambda^{(m)}|} Z_{\Lambda^{(m)}} \left( \{ c_j^L(m) \} \right) < Z_{\Lambda^{(m-1)}} < F^U_0(m)^{|\Lambda^{(m)}|} Z_{\Lambda^{(m)}} \left( \{ c_j^U(m) \} \right) \]

implies that

\[ \tilde{F}_0(m, \alpha, t)^{|\Lambda^{(m)}|} Z_{\Lambda^{(m)}} \left( \{ \tilde{c}_j(m, \alpha) \} \right) = Z_{\Lambda^{(m-1)}} \] (1)

for some value

\[ 0 < \alpha = \alpha^{(m)}_{\Lambda}(t) < 1 \]

**Note:** Parametrization invariance under shift in t.
So \( \alpha = \alpha^{(m)}_{\Lambda}(t) \) is the level surface fixed by (1).

\[ \alpha^{(m)}_{\Lambda}(t) = \alpha^{(m)}(t) + \delta \alpha^{(m)}_{\Lambda}(t) , \quad \delta \alpha^{(m)}_{\Lambda}(t) \to 0, \quad |\Lambda| \to \infty \]
So starting from original lattice one gets an *exact integral representation* of the PF on successively decimated lattices:

\[
\begin{align*}
Z_\Lambda(\beta) &= Z_\Lambda\left(\{c_j(\beta)\}\right) \\
&= \tilde{F}_0(1, \alpha^{(1)}_\Lambda(t_1), t_1)^{|\Lambda^{(1)}|} Z^{(1)}_\Lambda \left(\{\tilde{c}_j(1, \alpha^{(1)}_\Lambda(t_1))\}\right) \\
&= \cdots \\
&= \left[ \prod_{m=1}^{n} \tilde{F}_0(m, \alpha^{(m)}_\Lambda(t_m), t_m)^{|\Lambda|/b^{dm}} \right] Z^{(n)}_\Lambda \left(\{\tilde{c}_j(n, \alpha^{(n)}_\Lambda(t_n))\}\right)
\end{align*}
\]
\[ Z_{\Lambda}^{(-)} \] is the partition function with action on every plaquette in set \( \mathcal{V} \) shifted by a non-trivial element (‘twist’) of the group center. For \( SU(2) \): \( \tau = -1 \in Z(2) \).

For \( Z_{\Lambda}^{(-)} \) reflection positivity holds only in planes perpendicular to the directions in which the set \( \mathcal{V} \) carrying the twist is winding around the lattice. To have RP in all planes simply replace \( Z_{\Lambda}^{(-)} \) by \( Z_{\Lambda} + Z_{\Lambda}^{(-)} \).
The above development then can be carried through in this case also.

\[
Z_\Lambda + Z_\Lambda^{(-)} = \left[ \prod_{m=1}^{n} \tilde{F}_0(m, \alpha_\Lambda^{+(m)}(t_m), t_m)|\Lambda|/b^{dm} \right] \\
\cdot \left[ Z_\Lambda^{(n)} \left( \{ \tilde{c}_j(n, \alpha_\Lambda^{(+)}(t_n)) \} \right) + Z_\Lambda^{(-)} \left( \{ \tilde{c}_j(n, \alpha_\Lambda^{+(n)}(t_n)) \} \right) \right]
\]

- **Note:** Flux presence does not affect bulk free-energy contributions.

- As indicated by the notation: the values \( \alpha_\Lambda^{+(m)}(t) \) fixed at each successive step \( m \) in this representation are a priori distinct from the values \( \alpha_\Lambda^{(m)}(t) \) in the representation for \( Z_\Lambda(\beta) \).

\[
\alpha_\Lambda^{+(m)}(t) = \alpha^{(m)}(t) + \delta \alpha_\Lambda^{+(m)}(t) , \quad \delta \alpha_\Lambda^{+(m)}(t) \rightarrow 0, \quad |\Lambda| \rightarrow \infty
\]
The vortex free-energy is defined as:

$$\exp(-F_\Lambda(\beta)) = \frac{Z_\Lambda^{(-)}}{Z_\Lambda}.$$ 

One may now represent this ratio on successively decimated lattices by inserting our representations in the numerator and denominator in

$$\left(1 + \frac{Z_\Lambda^{(-)}}{Z_\Lambda}\right) = \frac{Z_\Lambda + Z_\Lambda^{(-)}}{Z_\Lambda}$$

At each decimation step one may use the independent invariance under the parametrizations shifts in the representations to explicitly cancel the bulk free energy pieces generated at that step.
Carry out $n$ decimation steps with $n$ sufficiently large. Then one ends up with

$$\frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}} = \frac{Z_{\Lambda}^{(-)}(\{\tilde{c}_j(n, \alpha_{\Lambda}^*(n))\})}{Z_{\Lambda}(\{\tilde{c}_j(n, \alpha_{\Lambda}^*(n))\})} \quad (I)$$

Here $\alpha_{\Lambda}^*(n)$ denotes $\alpha_{\Lambda}^{(n)}(t)$ at a particular $t = t^*$ such that $\alpha_{\Lambda}^{(n)}(t^*) = \alpha_{\Lambda}^{+(n)}(t^*)$.

Now the coefficients in terms of which these PF’s are computed are bounded by the upper bound coefficients:

$$\tilde{c}_j(n, \alpha_{\Lambda}^*(n)) \leq c_j^U(n) \rightarrow 0 \quad \text{for} \quad n \rightarrow \infty.$$

Hence one can compute $(I)$ in the strong coupling convergent expansion.
Confining behavior is the result for any initial $\beta$ since
\[ c_j^U(n) \rightarrow 0 \quad \text{for} \quad n \rightarrow \infty \]
for any initial $\beta$.

Fixing the resulting string tension $\kappa(\beta, n)$ implies a relation between $n$ and $\beta = 2/g^2$.

Now zero coupling $g = 0$ is a fixed point of the decimations. This implies that to reach any fixed value of the string tension (some given value of $c_j^U(n)$’s) requires
\[ n \rightarrow \infty \iff \beta \rightarrow \infty. \]

In other words one necessarily has
\[ g(a) \rightarrow 0 \quad \text{for} \quad a \rightarrow 0 \]
as an essentially qualitative feature of the decimation flow.
Summary

A framework was developed that utilizes explicitly computable decimations to constrain the behavior of the exact theory. This has many potential applications.

Exact integral representation of the PF with or without external flux on successively coarser lattices.

Use these to look at order parameters – confinement at $T = 0$ emerges for any initial coupling once the approximate bounding decimations possess this property (arXiv:0707.2179).

Extension to $SU(3)$ straightforward – same qualitative flow.