Simulating $U(1)$ Gauge Theory on Non-Commutative Spaces

I. NC $U(1)$ gauge theory

II. Wilson loops in $d = 2$: area-preserving diffeomorphisms

III. NC QED$_4$: the fate of the NC photon

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I. NC $U(1)$ gauge theory

- NC Euclidean plane

NC space coordinates in $d = 2$ :

$$[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu} = i\vartheta\epsilon_{\mu\nu}$$

\(\hat{x}_\mu\) : Hermitian operators

\(\vartheta\) : NC parameter, const.

imply spatial uncertainty

$$\Delta x_\mu \Delta x_\nu \sim \theta$$

(cf. event horizon of a strong gravitation centre), and non-locality.

UV/IR mixing of divergences

→ perturbation theory beyond one loop is mysterious.
• Lattice structure

Non-perturbative approach:

Imposing the operator identity

\[ \exp \left( i \frac{2\pi}{a} \hat{x}_i \right) = \hat{1} \]

yields a (fuzzy) lattice structure.

Periodicity over the Brillouin zone → lattice is also spatially periodic:

\[ \frac{1}{2a} \vartheta p_i \in \mathbb{Z} \]
Periodic $N \times N$ lattice

\[ \Rightarrow \quad \vartheta = \frac{1}{\pi} Na^2 \]

**Double Scaling Limit**

\[ \begin{align*}
a &\to 0 \\
N &\to \infty \end{align*} \quad \text{at} \quad Na^2 = \text{const.} \]

leads to a **continuous NC plane of infinite extent**.

*Simultaneous UV and IR limit.*
Return to ordinary coordinates $x_\mu$, if all fields are multiplied by $\star$-products:

$$\phi(x) \star \psi(x) := \phi(x) \exp \left( \frac{i}{2} \overleftarrow{\partial}_\mu \Theta_{\mu\nu} \overrightarrow{\partial}_\nu \right) \psi(x)$$

e.g. $[x_\mu, x_\nu]_\star = i\Theta_{\mu\nu}$

**U(1) Gauge Theory**

$$S[A] = \frac{1}{4} \int d^2x \ F_{\mu\nu} \star F_{\mu\nu} , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig(A_\mu \star A_\nu - A_\nu \star A_\mu)$$

$S$ is $\star$-gauge invariant

(self-interaction !)

Cannot be simulated in this form on the lattice ($\star$-unitary link variables)
Way out: equivalence to a matrix model on one point

**Twisted Eguchi-Kawai Model** (González-Arroyo/Okawa, '83)

\[ S_{\text{TEK}}[U] = -N\beta \sum_{\mu \neq \nu} Z_{\mu \nu} \text{Tr}[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger] \]

\( U_\mu \): unitary \( N \times N \) matrices, \( \beta \equiv 1/g^2 \)

Twist: \( Z_{21} = Z_{12}^* = \exp(2\pi in/N) \), here: \( n = \frac{N+1}{2} \rightarrow N \text{ odd} \)

**Morita equivalence** (identical algebras) Aoki et al., '99

\[ \text{TEK}_{N \to \infty} \iff \text{NC } U(1) \text{ gauge theory on infinite lattice} \]

**Refinement** (Ambjørn et al., '00):

\[ \text{TEK}_{N \text{ finite}} \iff \text{NC } U(1) \text{ gauge theory on } N \times N \text{ lattice} \]
Mapping back the matrix model term (Ishibashi et al., Gross et al., ’00)

$$W_{\mu\nu}(I \times J) := \frac{1}{N} Z_{\mu\nu}^{I,J} \text{Tr}[U_{\mu}^{I} U_{\nu}^{J} U_{\mu}^{\dagger} U_{\nu}^{\dagger}]$$

to the lattice defines the **NC Wilson loop**.

Note that $W_{\mu\nu} \in \mathbb{C}$, but the action is real
(both orientations summed over, $W_{\mu\nu} = W_{\nu\mu}^*$)

⇒ In this form, Monte Carlo simulations are possible!

(W.B./Hofheinz/Nishimura, ’02)

Scale : $\beta = 1/a^2$ from Gross-Witten area law in the planar limit
→ fix $N/\beta$ for Double Scaling
II. Wilson loops in $d = 2$:
area-preserving diffeomorphisms (APDs)

Pure YM theories on a \textit{commutative} plane: soluble thanks to APD invariance; $\langle\text{Wilson loop}\rangle$ only depends on (oriented) area.

NC $U(n)$: Perturbation theory to $O(g^4\theta^{-2})$ reveals sym. breaking down to $SL(2, \mathbb{R})$. (Ambjørn/Dubin/Makeenko ’04, Bassetto/De Pol/Torrielli/Vian ’05)

Non-perturbative test for NC $U(1)$ with squares, L-shapes, rectangles, stairs (W.B./Bigarini/Torrielli ’07)
\(| < W > |\) for various shapes surrounding the same area,

\[ \theta \equiv \frac{1}{\pi} N a^2 = 2.63 \].

Results for different volumes \((N a)^2\) and different \(a\) agree.

Area law at small (dimensional) area, but deviations beyond

→ shape dependence persists in the Double Scaling Limit.
Focus on the rectangles to check sym. under the APD subgroup $SL(2,R)$.

Here we fix $\theta = 1.63$ at different volumes and lattice spacings:
we see again DSL convergence with (minor) sym. breaking.

$\Rightarrow$ On the non-pert. level, the APD sym. breaks completely.
No hope for analytic solution. Simulations are crucial, as in 4d YM theory.
III. NC QED$_4$ : the fate of the NC photon

We consider again a NC plane $[\hat{x}_1, \hat{x}_2] = i\vartheta = \text{const.}$, plus $x_3, t : \text{commutative.}$

NC photon: $\Theta$-deformed dispersion relation?

1-loop calculations suggest the form \cite{Matusis/Susskind/Thoumbas, '00}

$$E^2 = \vec{p}^2 + \frac{C}{(p\Theta)^2}$$

Test with data from cosmic photons! General ansatz:

$$E = |\vec{p}| + \frac{E}{M} \quad \text{(} M : "\text{quantum gravity foam}"	ext{)}$$

E.g. different time of flight for 35 Gamma Ray Bursts

$\Rightarrow \quad M > 0.001 M_{\text{Planck}} \quad \text{(Ellis/Mavromatos/Nanopoulos/Sakharov/Sarkisyan, '04)}$
Bounds for $\vartheta$ in Nature? (Amelino-Camelia et al. ’98)

But: 1-loop perturbation theory: IR singularity is negative
(Landsteiner/Lopez/Tytgat, ’00, . . .)
IR instability, ill defined, to be cured by SUSY . . .

- **NC QED\textsubscript{4} revisited non-perturbatively** (W.B./Nishimura/Susaki/Volkholz, ’06)
  - comm. plane $\rightarrow L \times L$ lattice
  - NC plane $\rightarrow$ matrix model (TEK) $(N \approx L)$

First goal: search for physical scale $\rightarrow$ identify a Double Scaling Limit.

**Successful ansatz:**

$$a \propto 1/\beta \rightarrow \vartheta \propto N/\beta^2$$

(different from NC QED\textsubscript{2}, fine-tuned scaling at each $N$ differs slightly)
Double Scaling for the Wilson loops:  

(Here \( N/\beta^2 \equiv 20 \))

![Graphs showing Re (Wilson loops) vs. physical area for commutative plane, mixed plane, and NC plane.]

- **Commutative plane**: \( \langle W \rangle \in \mathbb{R} \) due to sym. in signs of \( x_3 \) and \( t \)
Order parameter for translation sym. in NC plane: open Polyakov line

- gauge invariant, carries momentum $p$

(Ishibashi/Iso/Kawai/Kitazawa '00)

$$P_\mu(n) = \mathcal{P} \exp_\star \left( ig \int_x^{x+\tilde{p}_\mu} A_\mu(\xi) \, d\xi \right)$$

$\mathcal{P}$: path ordering; $\exp_\star$: power series with $\star$-product; $\tilde{p}_\mu \equiv \Theta_{\mu\nu} p_\nu = na\hat{\mu}$: length

$P_1$ for $N = 15, 25, 35$  

Coupling: strong $\xrightarrow{\beta \approx 0.35}$ moderate $\xrightarrow{\beta \approx 0.35}$ weak  

hysteresis
Phase diagram: **Weak** ↔ **Moderate** ↔ **Strong coupling**  
(cf. talks by Ishikawa and Vairinhos)

Double Scaling Limit  \( \beta \sim \sqrt{N} \) always leads to the **broken** phase.

In this limit, we found stability of all observables that could be measured well.

**Broken phase could describe a stable cont. limit for the NC photon.**
Dispersion Relation

determined from exp. decay in the comm. plane $E(p = p_3)_{|p_1=p_2=0}$

sym. phase
consistent with
neg. IR divergence
("tachyonic" behaviour)

broken phase
IR stable
Nambu-Goldstone boson
(SSB of transl. sym.)
V. Conclusions

We studied $\text{QED}_2$, and $\text{QED}_4$ on spaces with a NC plane.

Discretised NC plane can be mapped onto a TEK matrix model; enables MC simulations (heat-bath after linearisation through auxiliary matrix field)

A Double Scaling Limit to a continuous, infinite NC space converges
→ non-pert. renormalisable.

• $\text{QED}_2$ :
  - small area: Wilson loops obey area law for all shapes
  - large area: complex phase for many shapes $= \left( \frac{\text{area}}{\vartheta} \right)$, corresponds to AB effect with $B = 1/\vartheta$ (cf. W.B./Hofheinz/Nishimura ’02)

At large area the **APD sym. is broken, including the SL(2,R) subgroup.**
→ Unlike 2d YM theory: not analytically soluble, but rich structure, can be explored numerically.
• **QED**$_4$:

**Double Scaling Limit** identified by matching of Wilson loops at various $N$. Other observables follow the same scaling law.

Open Polyakov line as order parameter for transl. invariance:

$$\begin{cases} 
\beta < 0.35 \text{ (strong coupling)} & \text{symmetric} \\
\text{intermediate} & \text{broken} \\
\text{large } \beta & \text{symmetric}
\end{cases}$$

Transition line intermediate–weak: $\beta_c(N) \propto N^2$

$\rightarrow$ **Double Scaling Limit** $\beta \propto \sqrt{N}$ leads to broken phase, IR stable.

• Here and in $\lambda \phi^4$ model: Strong IR effects due to short-range non-locality

$\Rightarrow$ UV/IR mixing persists as a non-perturbative effect in NC field theory.