$B_K$ on 2+1-flavor Iwasaki DWF Lattices
(focusing on $24^3 \times 64$ volume)

S. D. Cohen
for the RBC/UKQCD Collaborations

Columbia University $\Longrightarrow$ Jefferson Lab

XXV International Conference on Lattice Field Theory
1. The Kaon Bag Parameter
2. Computational Results from $24^3 \times 64$
3. Chiral Extrapolation
4. Lattice Bag Parameter in the Field
$B_K$ determines the accuracy to which $\epsilon_K$ measurements constrain the $CP$-violating elements of the CKM matrix.

$$\epsilon = \hat{B}_K \text{Im}\lambda_t \frac{G_F^2 f_K^2 M_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K} \times \left\{ \text{Re}\lambda_c \left[ \eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t) \right] \cdot \text{Re}\lambda_t \eta_2 S_0(x_t) \right\} e^{i\pi/4}$$
$B_K$ parametrizes mixing between neutral kaons due to weak interactions. At leading order in the Standard Model, this mixing is due to the exchange of two $W$ bosons:
$B_K$ parametrizes mixing between neutral kaons due to weak interactions. At leading order in the Standard Model, this mixing is due to the exchange of two $W$ bosons:
Definition of $B_K$ in Terms of the Mixing Matrix Element

$$B_K = \frac{\langle K^0|O_{LL}^{\Delta S=2}|K^0\rangle}{\frac{8}{3}f^2KM_K^2} \quad O_{LL}^{\Delta S=2} = (\bar{s}d)_L(\bar{s}d)_L$$

S. D. Cohen (Jefferson Lab)
Choosing an Optimal Method for Calculating $B_K$

Wall Method

$$B_P = \frac{4}{8/3 f^2 M} \frac{C_{wpw}(t_1,t,t_2)}{C_{ww}(t_1,t_2)}$$

Pseudo Method

$$B_P = \frac{2M^2 Z_A^2 V}{8/3 (m_1+m_2+2m_{res})^2} \frac{C_{wpw}(t_1,t,t_2)}{C_{wp}(t_1,t)C_{wp}(t_2,t)}$$

Axial Method

$$B_P = \frac{2V}{8/3} \frac{C_{wpw}(t_1,t,t_2)}{C_{wp}(t_1,t)C_{wp}(t_2,t)}$$
Choosing an Optimal Method for Calculating $B_K$

**Wall Method**

$$B_P = \frac{4}{3} f^2 M \frac{C_{wpw}(t_1,t,t_2)}{C_{ww}(t_1,t_2)}$$

**Pseudo Method**

$$B_P = \frac{2M^2 Z_A^2 V}{\frac{8}{3}(m_1+m_2+2m_{res})^2} \frac{C_{wpw}(t_1,t,t_2)}{C_{wp}(t_1,t)C_{wp}(t_2,t)}$$

**Axial Method**

$$B_P = \frac{2V}{\frac{8}{3}} \frac{C_{wpw}(t_1,t,t_2)}{C_{wp}(t_1,t)C_{wp}(t_2,t)}$$
Choosing an Optimal Method for Calculating $B_K$

Wall Method

$$B_P = \frac{4}{3} f^2 M \frac{C^{PO}_P(t_1, t, t_2)}{C^{PP}_w(t_1, t_2)}$$

Pseudo Method

$$B_P = \frac{2M^2 Z^2}{8/3 (m_1 + m_2 + 2m_{res})^2} \frac{C^{PO}_w(t_1, t, t_2)}{C^{PP}_w(t_1, t)C^{PP}_w(t_2, t)}$$

Axial Method

$$B_P = \frac{2V}{8/3} \frac{C^{PO}_w(t_1, t, t_2)}{C^{PA}_w(t_1, t)C^{PA}_w(t_2, t)}$$
We generate 2+1 flavor domain-wall fermion Iwasaki-gauge lattices using the RHMC algorithm on QCDOC machines.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L_s$</th>
<th>$c_1$</th>
<th>$m_s$</th>
<th>$m_l$</th>
<th>$16^3 \times 32$</th>
<th>$24^3 \times 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.13</td>
<td>16</td>
<td>-0.331</td>
<td>0.04</td>
<td>0.005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td></td>
<td></td>
<td>0.01</td>
<td></td>
<td></td>
<td>(90 run)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.02</td>
<td></td>
<td>(150 run)</td>
<td>(90 run)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.03</td>
<td></td>
<td>(150 run)</td>
<td></td>
</tr>
</tbody>
</table>
Parameters of the RBC/UKQCD Gauge Ensembles

Basic Parameters

We generate 2+1 flavor domain-wall fermion Iwasaki-gauge lattices using the RHMC algorithm on QCDOC machines.

\[
\begin{array}{cccc|cc}
\beta & 2.13 & c_1 & -0.331 & 16^3 \times 32 & 24^3 \times 64 \\
L_s & 16 & & & (90 \text{ run}) & (90 \text{ run}) \\
\hline
m_s & 0.04 & m_l & 0.005 & - & - \\
& & & 0.01 & (150 \text{ run}) & (90 \text{ run}) \\
& & & 0.02 & (150 \text{ run}) & - \\
& & & 0.03 & (150 \text{ run}) & - \\
\end{array}
\]
Parameters of the RBC/UKQCD Gauge Ensembles

Scale and Residual Mass

- Our scale may be determined from
  - Rho mass
  - Static quark potential
  - Pion decay constant
  - Omega mass \( a^{-1} = 1.72(3) \) GeV

- Strange Quark Mass
  - from kaon mass \( a_{\text{ms}} = 0.384(14) \)

- Light Quark Mass
  - from pion mass \( a_{\text{ml}} = 0.00140(6) \)

- Residual Mass
  - from midpoint pseudoscalar correlator: \( a_{\text{res}} = 0.00315(3) \)
Long Plateaux Available on the Large Lattices

Pseudoscalar Mass

$M$ Plateau on $24^3$, $m_s^{\text{sea}} = 0.04$, $m_l^{\text{sea}} = 0.005$

$M$ Plateau on $24^3$, $m_s^{\text{sea}} = 0.04$, $m_l^{\text{sea}} = 0.01$
Long Plateaux Available on the Large Lattices

The Bag Parameter

\[ B \text{ Plateau on } 24^3, m_s^{\text{sea}} = 0.04, m_l^{\text{sea}} = 0.005 \]

\[ B \text{ Plateau on } 24^3, m_s^{\text{sea}} = 0.04, m_l^{\text{sea}} = 0.01 \]
Numerical Results for $B_K$ on $24^3$

no form applied

$B_P$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01$
Ordinary NLO 2+1f Partially Quenched XPT form due to Sharpe and Van de Water

\[ B_P = B_0 \left[ 1 + \frac{1}{48\pi^2 f^2 M_K^2} \left( bM_K^4 + c(M_X^2 - M_Y^2)^2 + dM_K^2(2M_L^2 + M_S^2) + \text{logs} \right) \right] \]
Ordinary NLO 2+1f Partially Quenched XPT form due to Sharpe and Van de Water

$B_P$ 2+1f Partially Quenched Chiral Fit

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01$

$B_K = 0.556(8)$
Ordinary NLO 2+1f Partially Quenched XPT form due to Sharpe and Van de Water

\[ B_P \text{ 2+1f Partially Quenched Chiral Fit} \]

\[ 24^3, a_m^{\text{sea}} = 0.04, a_m^{\text{sea}} = 0.005 \]

\[ 24^3, a_m^{\text{sea}} = 0.04, a_m^{\text{sea}} = 0.01 \]

\[ B_K = 0.556(8) \]
Ordinary XPT inside Its Range of Validity
form due to Sharpe and Van de Water

\[ B_P \ 2+1f \ Partially \ Quenched \ Chiral \ Fit \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005 \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01 \]

\[ B_0 = 0.30(3) \]

S. D. Cohen (Jefferson Lab)
Ordinary XPT inside Its Range of Validity
form due to Sharpe and Van de Water

$B_P 2+1f$ Partially Quenched Chiral Fit

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.005$

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.01$

$B_0 = 0.30(3)$
Ordinary XPT inside Its Range of Validity
form due to Sharpe and Van de Water

\[ B_P \text{ 2+1f Partially Quenched Chiral Fit} \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005 \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01 \]

\[ B_0 = 0.30(3) \]
Linear Unitary Extrapolation

form could be due to Anybody

with \( M_X = M_L \) and \( M_Y = M_S \):

\[
B_P = \left( B_0 + \delta B M_S^2 \right) \left[ 1 + \left( b + \delta b M_L^2 \right) M_L^2 \right]
\]
Linear Unitary Extrapolation
form could be due to Anybody

with $M_X = M_L$ and $M_Y = M_S$:

$$B_P = \left( B_0 + \delta B M_S^2 \right) \left[ 1 + \left( b + \delta b M_S^2 \right) M_L^2 \right]$$
Chiral Extrapolation

Linear Unitary Extrapolation
form could be due to Anybody

$B_P$ Unitary MultiStrange Fit

$24^3, a m^\text{sea}_s = 0.04, a m^\text{sea}_l = 0.005$

$24^3, a m^\text{sea}_s = 0.04, a m^\text{sea}_l = 0.01$

$B_K = 0.582(10)$
Linear Unitary Extrapolation

form could be due to Anybody

\( B_P \) Unitary MultiStrange Fit

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005 \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01 \]

\[ B_K = 0.582(10) \]
Linear Unitary Extrapolation

form could be due to Anybody

$B_P$ Unitary MultiStrange Fit

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01$

$B_K = 0.582(10)$
NLO 2+1f Partially Quenched XPT + Some NNLO
S&VdW with extra terms

\[ B_P = B_0 \left[ 1 + \frac{1}{48\pi^2 f^2 M_K^2} \left( b M_K^4 + c (M_X^2 - M_Y^2)^2 + d M_K^2 (2M_L^2 + M_S^2) \ight. \ight. \\
\left. \left. + n_1 M_K^6 + n_2 M_K^4 (2M_L^2 + M_S^2) + n_3 M_K^2 (2M_L^2 + M_S^2)^2 \right) \ight. \\
\left. \left. + \text{logs} + \text{more logs} \right) \right] \]
NLO 2+1f Partially Quenched XPT + Some NNLO
S&VdW with extra terms

\[ B_P = B_0 \left[ 1 + \frac{1}{48\pi^2 f^2 M_K^2} \right. \]
\[ \left. \left( b M_K^4 + c (M_X^2 - M_Y^2)^2 + d M_K^2 (2M_L^2 + M_S^2) + n_1 M_K^6 + n_2 M_K^4 (2M_L^2 + M_S^2) + n_3 M_K^2 (2M_L^2 + M_S^2)^2 + \text{logs + more logs} \right) \right] \]
Chiral Extrapolation

NLO 2+1f Partially Quenched XPT + Some NNLO
S&VdW with extra terms

\[ B_P = B_0 \left[ 1 + \frac{1}{48\pi^2 f^2 M_K^2} \left( bM_K^4 + c(M_X^2 - M_Y^2)^2 + dM_K^2(2M_L^2 + M_S^2) \right. \right. \]
\[ + n_1 M_K^6 + n_2 M_K^4(2M_L^2 + M_S^2) + n_3 M_K^2(2M_L^2 + M_S^2)^2 \]
\[ \left. \left. + \text{logs} + \text{more logs} \right] \right] \]
NLO 2+1f Partially Quenched XPT + Some NNLO S&VdW with extra terms

$B_P$ (Partially) NNLO 2+1f Partially Quenched Chiral Fit

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01$

$B_K = 0.552(10)$
NLO 2+1f Partially Quenched XPT + Some NNLO
S&VdW with extra terms

$B_P$ (Partially) NNLO 2+1f Partially Quenched Chiral Fit

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.005$

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.01$

$B_K = 0.552(10)$
Heavy-Strange 2f Partially Quenched XPT form due to Sharpe and Zhang

\[ B_P = \frac{B_0}{\sqrt{M_K}} \left[ 1 + \frac{1}{16\pi^2 f^2} \left( cM_X^2 + dM_L^2 - \log \right) \right] \]
Heavy-Strange 2f Partially Quenched XPT
form due to Sharpe and Zhang

$B_P$ 2+1f Partially Quenched Heavy–Strange Chiral Fit

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005$

$24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01$

$B_K = 0.568(10)$
Heavy-Strange 2f Partially Quenched XPT
form due to Sharpe and Zhang

\[ B_K = 0.568(10) \]
Heavy-Strange 2f Partially Quenched XPT
form due to Sharpe and Zhang

$B_P$ 2+1f Partially Quenched Heavy–MultiStrange Chiral Fit

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.005$

$24^3$, $a m_s^{\text{sea}} = 0.04$, $a m_l^{\text{sea}} = 0.01$

$B_K = 0.568(10)$
Heavy-Strange 2f Partially Quenched XPT form due to Sharpe and Zhang

\[ B_P \text{ 2+1f Partially Quenched Heavy–MultiStrange Chiral Fit} \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.005 \]

\[ 24^3, a m_s^{\text{sea}} = 0.04, a m_l^{\text{sea}} = 0.01 \]

\[ B_K = 0.568(10) \]
Summary of $B_K$ from Various Chiral Forms

With $Z_{B_K} = 0.919(07)(26)$:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(3) NLO XPT</td>
<td>0.511(09)(14)</td>
</tr>
<tr>
<td>Unitary</td>
<td>0.536(10)(15)</td>
</tr>
<tr>
<td>SU(3) NLO + some NNLO</td>
<td>0.507(10)(14)</td>
</tr>
<tr>
<td>SU(2) NLO XPT</td>
<td>0.522(10)(15)</td>
</tr>
</tbody>
</table>

* S. Li’s talk, Mon 15:00
Comparison of Our $B_K$ to Other Lattice Calculations

Comparison of Lattice QCD Measurements of $B_K$

$B_K^\overline{\text{MS}}(2 \text{ GeV}) = 0.522(10)(15)$
Comparison of Lattice QCD Measurements of $B_K$

$B_K^{\overline{MS}}(2 \text{ GeV}) = 0.522(10)(15)$
Comparison of Our $B_0$ to Phenomenological Models

$B_0 = 0.28(5)$

<table>
<thead>
<tr>
<th>Model</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLO Large-$N_c$</td>
<td>0.29(15)</td>
</tr>
<tr>
<td>QCD-Hadron Duality</td>
<td>0.39(10)</td>
</tr>
<tr>
<td>Sum Rules</td>
<td>0.55(25)</td>
</tr>
</tbody>
</table>

![Diagram showing the comparison of $B_0$ values with different models.]
Application of Our $B_K$ to the Unitarity Triangle

\[ \epsilon = \hat{B}_K \text{Im} \lambda_t \frac{G_F^2 f_K^2 M_K M_W^2}{6 \sqrt{2} \pi^2 \Delta M_K} \times \left\{ \text{Re} \lambda_c \left[ \eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t) \right] \right\} \left( \text{Re} \lambda_t \eta_2 S_0(x_t) \right) e^{i \pi/4} \]

\[ B_K^{\text{MS}} (2 \text{ GeV}) = 0.522(10)(15) \]
Application of Our $B_K$ to the Unitarity Triangle

$$\epsilon = \hat{B}_K \text{Im}\lambda_t \frac{G_F^2 f_K^2 M_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K} \times \left\{ \text{Re}\lambda_c \left[ \eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t) \right] \right\} \cdot \text{Re}\lambda_t \eta_2 S_0(x_t) \cdot e^{i\pi/4}$$

$$B_K^{\text{MS}}(2 \text{ GeV}) = 0.522(10)(15)$$
Conclusions

- 2+1-flavor DWF determination of the kaon bag parameter at this lattice spacing is complete:
  \[ B_K^{\text{MS}}(2 \text{ GeV}) = 0.522(10)_{\text{stat}}(15)_{\text{ren}}(??)_{\text{chiral}}(??)_{\text{FV}}(30)_{\text{scaling}} \]

- Ordinary SU(3) chiral perturbation theory has substantial error at the mass of the kaon.

Outlook

- D. Antonio’s discussion of 16^3 and systematic errors in the very near future.
- Calculation on finer lattices already underway.