A QCD critical point at small chemical potential: is it there or not?

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with
Seyong Kim (U. Sejong) and Owe Philipsen (U. Münster)
Recall: QCD phase diagram vs \((m_u, d, m_s), T\) and \(\mu\)

\[\mu = 0\]

Now turn on \(\mu\)
Recall: QCD phase diagram vs \((m_{u,d}, m_s), T\) and \(\mu\)

\[\mu \neq 0\]

Conventional wisdom: first-order region expands with real \(|\mu|\)
Recall: QCD phase diagram vs \((m_u, d, m_s, T, \mu)\)

Exotic scenario: first-order region shrinks with real \(\mu\) \(\frac{d m_c}{d\mu^2} \big|_{\mu=0} < 0\)

QCD critical point DISAPPEARED

Real world
Heavy quarks

crossover

\(\mu\)

0

\(m_{u,d}\)

\(\infty\)

\(m_s\)
Recall: QCD phase diagram vs \((m_u, m_d, m_s), T\) and \(\mu\)

For heavy quarks, first-order region shrinks \((PdF, Kim, Takaishi, hep-lat/0510069)\)
Recall: QCD phase diagram vs \((m_u, m_d, m_s), T\) and \(\mu\)

Also Standard Model at finite lepton density (Gynther, hep-ph/0303019)
Lattice study with Owe Philipsen (hep-lat/0607017)

Strategy: tune $m_q$ for 2nd-order P.T. at $\mu = 0$, then turn on [imaginary] $\mu$

Does the transition become 1rst-order (left) or crossover (right)?
Motivation

Potts QCD

Lattice study with Owe Philipsen (hep-lat/0607017)

Strategy: tune $m_q$ for 2nd-order P.T. at $\mu = 0$, then turn on [imaginary] $\mu$

Does the transition become 1rst-order (left) or crossover (right)?

Answer: very little change ($\rightarrow$ surface almost vertical); crossover favored ($2\sigma$)

$$\frac{m_c(\mu)}{m_c(0)} = 1 - 0.7(4) \left( \frac{\mu}{\pi T} \right)^2$$

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**Binder cumulant:** \( B_4 = \langle (\delta \bar{\psi} \psi)^4 \rangle / \langle (\delta \bar{\psi} \psi)^2 \rangle^2 \)

**Thermodynamic limit:**
1 (1st-order) 3 (crossover) 1.604.. (3d Ising 2nd-order)

**Taylor expand:**
\[
B_4(am, a\mu) = 1.604 + b_{10} \left[ am - am_0^c - c'_1(a\mu)^2 \right] \\
+ b_{20} (am - am_0^c)^2 - b_{10} \left[ (c'_2 - c'_1 C)(a\mu)^4 + C(am - am_0^c)(a\mu)^2 \right]
\]

**Exotic scenario? → Is \( c'_1 \) negative?**

<table>
<thead>
<tr>
<th>( m_0^c )</th>
<th>( b_{10} )</th>
<th>( b_{20} )</th>
<th>( c'_1 )</th>
<th>( c'_2 )</th>
<th>( C )</th>
<th>( \chi^2 )/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0262(7)</td>
<td>13.3(1.4)</td>
<td>-91.6(143.5)</td>
<td>-0.079(47)</td>
<td>-1.6(1.0)</td>
<td>-2.1(3.5)</td>
<td>0.90</td>
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<tr>
<td>0.0263(6)</td>
<td>13.9(0.6)</td>
<td>—</td>
<td>-0.075(42)</td>
<td>-1.35(0.73)</td>
<td>—</td>
<td>0.82</td>
</tr>
<tr>
<td>0.0270(5)</td>
<td>13.6(0.6)</td>
<td>—</td>
<td>-0.0024(160)</td>
<td>—</td>
<td>—</td>
<td>0.93</td>
</tr>
<tr>
<td>0.0271(3)</td>
<td>13.6(0.6)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.88</td>
</tr>
</tbody>
</table>
How to do better?

Goal:
measure derivatives $\frac{dB_4}{d(am)}$ and $\frac{dB_4}{d(a\mu)^2}$ at the critical point $am_c(\mu = 0)$

Two strategies:
- express derivatives as vev of non-local operators (cf. Bielefeld-Swansea)
- measure change $\delta B_4$ under tiny changes in $m$ and imaginary $\mu$
  by reweighting $\mu = 0$ ensemble

Advantages of ●
- no sign pb. (imaginary $\mu$) or overlap pb. (tiny $\mu$)
- statistical fluctuations cancel between original and reweighted ens.
- reweighting factor $\frac{\det(\varphi(m+\delta m,\delta \mu))}{\det(\varphi(m,\mu=0))}$ can be estimated $\rightarrow$ cheap
- much simpler to implement than ●
**Heavy dense quarks → Potts model (hep-lat/0510069)**

- static dense QCD: $Z(\mu) = \int \mathcal{D}U \exp(-S_g[U] + e^{-\frac{M}{T}} + \frac{\mu}{T} \Phi[U] + e^{-\frac{M}{T}} - \frac{\mu}{T} \Phi[U]^{*})$

- Simplify gauge dynamics → Potts interaction (spin $\Phi_i \in \mathbb{Z}_3$):
  $$H = -k \sum_{ij} \delta_{\Phi_i, \Phi_j} - \sum_i [h\Phi_i + h'\Phi_i^{*}], \quad h = \exp(-\frac{M}{T} + \frac{\mu}{T}); \quad h' = \exp(-\frac{M}{T} - \frac{\mu}{T})$$

---

**Potts phase diagram**

2nd-order line: $\frac{M}{T} = 8.273 + 0.585 \left(\frac{\mu}{T}\right)^2 - 0.174 \left(\frac{\mu}{T}\right)^4 + 0.160 \left(\frac{\mu}{T}\right)^6 - 0.071 \left(\frac{\mu}{T}\right)^8$

Reproduce $\uparrow$ by reweighting from $\mu = 0$ ?
Reweighting critical $\mu = 0$ Potts ensemble

in $h = e^{-\frac{M}{T}}$

$\delta B_4 = \frac{d B_4}{d M/T} \delta \frac{M}{T} + \frac{d B_4}{d (\mu/T)^2} \delta (\frac{\mu}{T})^2$

$\delta B_4 = 0 \implies \frac{\delta M/T}{\delta (\mu/T)^2} = -\frac{d B_4}{d (\mu/T)^2} \frac{d B_4}{d M/T} \rightarrow 0.589(8)$ vs $0.585(3)$ from global $\mu$ fit
First, check curvature of pseudo-critical line, ie. \( \frac{d\beta_c}{d(\mu^2)^2} \):

\text{hep-lat/0607017}: 0.781(7) (\mu^2 \text{ fit}) \text{ or } 0.759(22) (\mu^2 + \mu^4 \text{ fit})
First, check curvature of pseudo-critical line, ie.

\[ \frac{d\beta_c}{d(a\mu)^2} : \]

reweight \( \mu = 0 \) ensembles to \( a\mu_I = 0.01 \cdots 0.1 \)

- consistent with old \( (\mu^2 + \mu^4) \) fit \( \rightarrow \) systematic error \( \approx \) removed
First, check curvature of pseudo-critical line, ie. \( \frac{d\beta_c}{d(a\mu)^2} \): reweight \( \mu = 0 \) ensembles to \( a\mu_I = 0.01 \cdots 0.1 \)

- consistent with old \( (\mu^2 + \mu^4) \) fit \( \rightarrow \) systematic error \( \approx \) removed
- statistical error reduced by \( O(10) \) for \( O(1/10) \) the computer effort!

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Now, $B_4(\am, a\mu) = 1.604 + b_{10} \left[ \am - \am_0^c - c'_1(a\mu)^2 \right]$

- $\frac{dB_4}{d(\am)} = b_{10} \sim 13.6(6)$ well determined from hep-lat/0607017
- $c'_1$ poorly determined, with $\mu^2$ and $(\mu^2 + \mu^4)$ fits
Back to $N_f = 3$ QCD (2)

Now, $B_4(am, a\mu) = 1.604 + b_{10} \left[ am - am_0^c - c_1'(a\mu)^2 \right]$

- $\frac{d B_4}{d (am)} = b_{10} \sim 13.6(6)$ well determined from hep-lat/0607017
- $c_1'$ poorly determined, with $\mu^2$ and $(\mu^2 + \mu^4)$ fits

$c_1' \sim -0.09(1)$, negative beyond doubt
Back to $N_f = 3$ QCD (2)

Now, $B_4(am, a\mu) = 1.604 + b_{10} [am - am_0^c - c'_1 (a\mu)^2]$

- $\frac{dB_4}{d(am)} = b_{10} \sim 13.6(6)$ well determined from hep-lat/0607017
- $c'_1$ poorly determined, with $\mu^2$ and $(\mu^2 + \mu^4)$ fits

\[ dB_4/d(\mu^2) \]

$\delta B_4 \approx 0.012$

$c'_1 \sim -0.09(1)$, negative beyond doubt
Conclusions

- First-order region shrinks when $\mu$ is turned on for $N_f = 3$
  with $N_t = 4$ staggered quarks ($a \sim 0.3$ fm)

- First-order region shrinks when $a \to 0$
  (Bielefeld, MILC; talks by O. Philipsen, G. Endrodi)
  $\rightarrow$ physical critical surface moves away from physical point

**QCD critical point at small $\mu$ (ie. $\frac{\mu}{T} < 1$) unlikely**

- Numerical method: 4 CPU-years of Itanium,
  or 4 days of Grid computing
  (thanks to Massimo Lamanna, CERN)
Resulting phase diagram (simplest possibility)

Standard scenario

Exotic scenario

$T_c$
Resulting phase diagram (simplest possibility)

**Standard scenario**

- $m < m_c(0)$
- $T_{c}$
- Confined
- QGP
- Color superconductor

**Exotic scenario**

- $m << m_c(0)$
- $T_{c}$
- Confined
- QGP
- Color superconductor

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Exotic scenario

\( m > m_c(0) \)

T

\( \mu \)

\( m_s \)

\( m_u \)

\( m_c \)

Confined

Color superconductor

QGP

\( T_c \)

\( N_f = 1 \)

\( N_f = 2 \)

\( N_f = 3 \)

2nd order

Z(2)

2nd order

Z(2)

1st order

O(4)

Crossover

Pure Gauge

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$m > m_c(0)$
Resulting phase diagram (simplest possibility)

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Reweighting with noisy estimator of reweighting factor

Reweighting: $\langle W \rangle_0 = \frac{1}{N} \sum_i W_i \rightarrow \langle W \rangle_\rho = \frac{\sum_i \rho_i W_i}{\sum_i \rho_i}$

Add unbiased noise to $\rho_i$: $\rho_i \rightarrow \rho_i + \eta_i$, $\langle \langle \eta_i \rangle \rangle = 0$

Here, $\rho = \frac{\det(D(\mu=i\mu_I))}{\det(D(\mu=0))} = \langle \langle \exp(-|D(i\mu_I)^{-1/2}D(0)^{1/2}\eta|^2 + |\eta|^2) \rangle \rangle$

No difficulty: $\langle W \rangle_\rho = \frac{\langle \langle \sum_i \rho_i W_i \rangle \rangle}{\langle \langle \sum_i \rho_i \rangle \rangle}$, eg. by jackknife

Careful with $\langle W \rangle_\rho^2$: $\langle \langle \eta_i \eta_j \rangle \rangle = 0$ only if $i \neq j$

Small effect on peak of susceptibility:

![Graph showing small effect on peak of susceptibility]

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<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$m_H^{\text{end point}}$</th>
<th>$T_c^{\text{end point}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 GeV</td>
<td>72 GeV</td>
<td>109 GeV</td>
</tr>
<tr>
<td>15 GeV</td>
<td>71 GeV</td>
<td>108 GeV</td>
</tr>
<tr>
<td>30 GeV</td>
<td>66 GeV</td>
<td>104 GeV</td>
</tr>
<tr>
<td>45 GeV</td>
<td>52 GeV</td>
<td>94 GeV</td>
</tr>
</tbody>
</table>

\[ \frac{m_H(\mu)}{m_H(\mu=0)} \approx 1 - 16 \left( \frac{\mu}{\pi T} \right)^2 \]