Scaling of hadronic observables in QCD with $N_f = 2$ maximally twisted Wilson quarks

Roberto Frezzotti
University and INFN of Roma Tor Vergata

Lattice 2007 - Regensburg - Jul. 30th–Aug. 4th
Introduction – I

Goal of the talk: “summarise” info about scaling from 2 + “1/2” lattice spacings

Omit $m^2_{\pi 0}$: special case with large $O(a^2)$; other flavour-breaking $O(a^2)$ are tiny
(→ talks by C. Michael and G.C. Rossi)

Action: tree-level improved Symanzik glue & $N_f=2$ Wilson-tm flavours

$$S_F = a^4 \sum_x \bar{\chi}(x)[\gamma \nabla i - a/2 \nabla^* \nabla + m_0 + i \mu \gamma_5 \tau^3] \chi(x)$$

$m_R, \mu_R$ and $M_R = \sqrt{m_R^2 + \mu_R^2}$

$\mu_R = Z_P^{-1} \mu$ and $m_R = Z_P^{-1} Z_A m_{PCAC}$, $m_0 \leftrightarrow m_{PCAC}$ extracted from lattice WTI

Maximal twist: $m_R = 0, \mu_R = O(a^0)$ as $a \to 0$

$$S_F = a^4 \sum_x \bar{\psi}(x)[\gamma \nabla i - i \gamma_5 \tau^3 (-a/2 \nabla^* \nabla + m_{cr}) + \mu] \psi(x), \quad \psi: \text{“physical” basis}$$

- generic: $m_0 = m_{cr}$, ambiguity $O(a \Lambda^2, a \mu \Lambda)$

- optimal: $m_0 = m_{cr}^{PCAC}(\mu)$, ambiguity $O(a \mu \Lambda)$

- optimal–practical: $m_0 = m_{cr}^{PCAC}(\mu_{LOW})$ and $\mu \gtrsim \mu_{LOW}$
Mtm-Wilson quarks: advantages and drawbacks (talks by C. Urbach, G.C. Rossi)

+ (ultra)local → renormalizability, predictivity
+ numerically cheap & robust IR cutoff in place (mult. renormalized $\mu$)
+ all physical observables automatically $O(a)$ improved (by symmetry)

- reliable in window $\mu \gtrsim \mu_{\text{LOW}} \sim a^2 \Lambda^3$ and $\mu \ll 1/a$ because
  
  $\star$ $V = \infty$: lattice–peculiar phase transition near to the massless “point”
  
  $\star$ $V = (3\text{fm})^4$: long fluctuations in simulations near to the massless “point”

most evident in observables ($m_{\text{PCAC}}$, plaq.) discontinuous at transition
survive as $V \to \infty$: additional to those seen at small $V$ (Del Debbio et al ‘05)

- an estimate of $m_{\text{cr}}(g_0^2)$ precise up to $O(a\Lambda^2)$ is necessary to avoid numerically large $O(a^2)$ at small masses (Frezzotti et al’ 05). In practice:

\begin{align*}
\mu & \gtrsim \mu_{\text{LOW}} \sim a^2 \Lambda^3 \quad \text{(few MeV if } a^{-1} \sim 2\text{GeV)} \\
\varepsilon \equiv \text{(numerical error on ”} m_{\text{PCAC}} = 0\text{”)} & \ll \mu_{\text{LOW}}
\end{align*}
Optimal estimate of $m_{cr}$ - Control of $O(\alpha^2)$

(A) generic $m_0 = m_{cr}$: left-over relative artifacts \( (\xi_\pi / \mu \Lambda^2)^2 \sim \alpha^2 / \mu^2 \)

\[
\xi_\pi = \xi_\pi(\mu) = |\langle \Omega | aL^\text{Mtm-Sym}_5 + O(\alpha^3) | \pi^0(0) \rangle|_\mu^\text{cont}
\]

\[
L^\text{Mtm-Sym}_5 = \eta_{SW} 1/4 \bar{\psi} [\sigma F] \gamma_5 \tau^3 \psi + \eta_{\text{soft}} \mu^2 \bar{\psi} i \gamma_5 \tau^3 \psi + \eta_{\text{NP}} \Lambda^2_{\text{QCD}} \bar{\psi} i \gamma_5 \tau^3 \psi
\]

(B) $m_0 = m_{cr}^{PCAC}(\mu) + O(\varepsilon)$: obtained via $m_{PCAC}(\mu) = 0 \pm \varepsilon$

left-over relative artifacts \( (a \Lambda + \varepsilon / \mu + a \Lambda \frac{\alpha^2 \Lambda^3}{\mu} + a \mu)^2 \)

(C) $m_0 = m_{cr}^{PCAC}(\mu_{LOW}) + O(\varepsilon)$ and limit work to $\mu \gtrsim \mu_{LOW}$

left-over relative artifacts \( (a \frac{\mu_{LOW}}{\mu} + a \Lambda + \frac{\varepsilon}{\mu} + a \Lambda \frac{\alpha^2 \Lambda^3}{\mu} + a \mu)^2 \)

(A) leaves too large $O(\alpha^2)$, (B) needs tuning at each $\mu$ (unpractical)

we choose (C) $\Rightarrow$

- small $O(\alpha^2) \Leftrightarrow \mu \gtrsim \text{coefficient} \times \alpha^2 \Lambda^3 \gg \varepsilon$ (coefficient from simulations)
- left-over artifacts non monotonic in $\mu$: grow as $\mu \rightarrow 0$ (due to $\mu_{LOW}, \varepsilon$)
- uniform $O(\alpha^2) \Leftrightarrow$ roughly the same $\mu_{LOW}$ for all lattice spacings
Implementation and checks of maximal twist

Simulations at $\beta = 4.05, 3.9, 3.8$ corresponding to $a \simeq 0.067, 0.086, 0.100$ fm

$m_R r_0 = Z_P^{-1} Z_A m_{PCAC} r_0$ vs $\mu_R r_0 = Z_P^{-1} \mu r_0$ : check smallness of $\varepsilon/\mu \leftrightarrow m_R/\mu_R$

lowest $\mu_R(M_S, 2 GeV) \simeq 20, 20, 32$ MeV for $\beta = 4.05, 3.9, 3.8$

Deviations from criteria for maximal twist

- negligible @ $\beta = 4.05, 3.9$: same $\mu_{LOW}$ and $m_R/\mu_R|_{\mu_{LOW}} \ll 1$ consistent with 0
- significant @ $\beta = 3.8$: $\tau_{int}[am_{PCAC}] = O(100)$ for $\mu_R r_0 < 0.1 \Rightarrow$ statistical errors?
Renormalization conditions

\[ r_0^2 F_{Q\bar{Q}}(r_0) = 1.65 \text{ (Sommer scale)} \Rightarrow r_0 / a \leftrightarrow g_0^2 \]
\[ m_{PS} r_0 = \text{constant} \ (0.7, 0.8, 0.9, 1.0, 1.1, 1.25) \Rightarrow a m_{PS} \leftrightarrow a \mu \]

Continuum limit: \( a / r_0 \rightarrow 0 \), at fixed spatial volume \( L_{\text{ref}}^3 = (2.2 \text{ fm})^3 \)

Data brought from \( L \in [2.0, 2.4] \text{ fm} \) to \( L_{\text{ref}} \): maximum relative change is 0.7%

hep-lat/0503014 (Colangelo et al.) for FSE’s \( (N^{1,2}\text{LO ChPT + resummed Lüscher formula}) \)

Interpolation in \( m_{PS}^2 r_0^2 \) to match \( m_{PS} r_0 = \# \): compare polynomial & chiral fits

Estimate continuum limit by average of results from \( \beta = 3.9 \) and \( \beta = 4.05 \)

systematic error: difference between this estimate and the result from \( \beta = 3.8 \)

Extra conditions for partially quenched observables involving \( D \) and \( D_s \) mesons:
\[ m_D r_0 \rightarrow \text{plot’s x-axis} \quad \& \quad \mu_s / \mu_c = 0.082 \quad (\text{“close-to-realistic”}) \]
The evaluation of $r_0$: some info

HYP-smeared temporal links (Della Morte et al. ’05), APE-smeared spatial links interpolating operators diff. by spatial smearing level $\Rightarrow$ correlator matrix

generalized eigenvalue problem $\Rightarrow$ ground and (roughly) first excited states

In scaling test: use the values of $r_0/a = r_0/a(\mu^2)$ extrapolated to $\mu = 0$

$\lim_{\mu \to 0} r_0/a[\mu^2] : \quad 6.61(3), 5.22(2), 4.46(3)$ for $\beta = 4.05, 3.9, 3.8$
Simulation data: blue points; weighted average of $\beta = 3.9$ and 4.05: red line
A first estimate of continuum limit result and errors
+ red symbol @ $a = 0$: central value with merely statistical errorbar
- green symbol @ $a \lesssim 0$: estimated systematic error, using the result at $\beta = 3.8$
+ green symbol @ $a > 0$: “data brought to max. twist” (shown if correction $> 1\sigma$)
About “bringing data to maximal twist”

Amounts to “correcting” the $O(a^0)$ part of the error in imposing $m_{PCAC} = 0$. Consider the case: $m_R = O(\epsilon)$ non-negligible wrt. $\mu_R$, due to numerical errors

$$\tan \theta = m_R / \mu_R$$

to be treated as $O(a^0)$  \quad $\theta = \pi/2$ “twist angle” $\Rightarrow$

- the renormalized quark mass is $M_R = (\mu^2_R + m^2_R)^{1/2} = \mu_R / \cos \theta$
- all operators formally non–invariant under axial-$\tau^3$ transformations must be reinterpreted consistently with “twist angle” $\alpha = \pi/2 - \theta$, e.g.

$$Z_V \bar{\chi} \gamma_\mu \tau^2 \chi = \cos \theta A^1_{\mu, R} + \sin \theta V^2_{\mu, R}$$

with $A^b_{\mu, R}$, $b = 1, 2, 3$ axial current...

- \[ < \pi^1 | Z_V \bar{\chi} \gamma_\mu \tau^2 \chi | \Omega > | m_R, \mu_R = \cos \theta < \pi^1 | A^1_{\mu, R} | \Omega > | M_R > | \text{cont} + O(a\theta, a^2) \]

hence  \[ f_{\pi^\pm} | m_R, \mu_R = \cos \theta f_{\pi^\pm} | M_R > | \text{cont} + O(a\theta, a^2) \]

Peculiar $O(a\theta, a^2)$: “corrected results” at $\beta = 3.8$ hardly usable for $a \to 0$ limit

No evidence for large cutoff effects at $\beta = 3.8$ provided maximal twist is precisely implemented – difficult if $\tau_{\text{int}}(am_{PCAC}) \gtrsim O(100)$, see Urbach’s talk.
\( \mu_R \) vs \( \alpha^2 \) for fixed \( m_{PS} \) values

Check of universality (\( \beta \)-independence) of \( m_{PS} = m_{PS}(\mu_R, \alpha_R(L_{\text{ref}}^{-1})) \)

matching \( m_{PS} r_0 \) and \( L_{\text{ref}}/r_0 \) at various \( \beta \)'s ⇔ matching \( \mu_R r_0 \) – up to \( \mathcal{O}(\alpha^2) \)

the scheme- and scale-independent ratios \( Z_P^R(qa; \beta)/Z_P^R(qa'; \beta') \) are estimated from the ratio of \( \mu|_{\beta; L_{\text{ref}}, m_{PS}} \) to \( \mu|_{\beta'; L_{\text{ref}}, m_{PS}} \), for \( m_{PS} r_0 = 1 \)

Plot \( \mu_R(\overline{MS}, \text{2 GeV}) \) obtained from \( Z_P^{RI-MOM}(1; \beta = 3.9) \) via NL^3O-evolution:

ease intuition; scaling is independent from overall scheme-dependent \( Z_P^{-1} \)
$m^V$ and $m_N$ vs $m_{PS}^2$

$L = \infty$: $\rho$-state unstable as $m^V > 2m_{PS}$. Here $L \sim 2.2$ fm: $\rho$–$\pi\pi$ mixing? ⇒ "$m^V$

good scaling for all three $\beta$’s, though with “largish” stat. errors at small $\mu$

nucleon mass $m_N$: in progress; $\beta = 3.9$ and lowest $\mu'$s of $\beta = 4.05$ promising
Charmed observables & PQ setup: $f''_D$

close-to-realistic ren. conditions: $m_{PS} r_0 = 0.7092$ and $\mu'' s'' / \mu'' c'' = 0.082$

(instead of the realistic conditions $m_{\pi} r_0 \simeq 0.30$ and $\mu_s / \mu_c \simeq 0.088$)

cutoff effects at most of natural size $a^2 \mu''^2 s'', c'': \leq 10\%$ (see next slide)

recall: $a\mu'' s'' \sim 0.020 \div 0.033$ and $a\mu_c \sim 0.25 \div 0.40$ ($\mu_c$ realistic)
Charmed observables & PQ setup: $f_{D_S}$, $m''_{D_S}$

$\mu_s / \mu_c = 0.082$ and $m_{\pi^0} r_0 = 0.7092$

$O(a)$ improved charmed observables: in general need continuum limit
three/four lattice spacings may allow to control residual $O(a^2)$ at the 1–2% level
Scaling of all observables very satisfactory, consistent with $O(a)$ improvement:

- $\pi^{\pm}$ channel, $\rho^{\pm}$ and nucleon mass, $r_0$ (static $Q\bar{Q}$ potential), quark mass
- charmed observables in PQ setup: "$D$" and "$D_s$" channels
- for $\beta = 4.05$ and $\beta = 3.9$ corresponding to $a \simeq 0.067$ fm and $a \simeq 0.086$ fm
- also for $\beta = 3.8$ ($a \simeq 0.100$ fm): $\pi^{\pm}$- and $\rho^{\pm}$-channel results reliable when data "corrected to max. twist" coincide with raw data (in spite of the long-range fluctuations in $m_{\text{PCAC}} @ \mu_{\text{LOW}}$ & imperfect tuning to max. twist)

Tuning $m_0$ to optimal estimate of $m_{\text{cr}}$ at $\mu = \mu_{\text{LOW}}$ is important: requires care & precision; can be tested by checking stability under "corrections" to max. twist.

Flavour-breaking $O(a^2)$ effects very small, with a most remarkable exception: $m^2_{\pi_0}$, understood (related to $a^2 | < \Omega | P | \pi >^{\text{cont}} |^2$, with $| < \Omega | P | \pi >^{\text{cont}} |$ large)