The phase structure of a chirally invariant Higgs-Yukawa model

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Organization of the talk

- 1. Introduction and motivation
- 2. Phase structure at weak Yukawa coupling
  → Analytical large $N_f$-limit vs. Numerical results
- 3. Phase structure at strong Yukawa coupling
  → Analytical large $N_f$-limit vs. Numerical results
- 4. Preliminary results on upper Higgs mass bound
- 5. Outlook
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Results published in

<table>
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<th>arXiv: 0705:2539</th>
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<tr>
<td></td>
<td>Julius Kuti</td>
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<td>arXiv: 0707:3849</td>
<td>Kieran Holland</td>
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1.1 Introduction and motivation

- LHC will explore Higgs sector soon.
  - Theoretical predictions on Higgs properties are of particular interest now.
- Available theo. Higgs mass bounds depend strongly on perturbation theory.
  - Concerns that at least lower bound, based on vacuum instability, is fake (Kuti, Holland)
- Non-perturbative determination of Higgs mass bounds desired.
  - Study pure Top-Higgs sector with Higgs-Yukawa models.
- Earlier Higgs-Yukawa models explicitly broke chiral symmetry.
  - In continuum limit chiral symmetry restoration and lifting of fermion doublers could not be achieved simultaneously.

Study Higgs-Yukawa model with built-in chiral symmetry.
1.2 The model

- A chirally invariant $SU(2)_L \times SU(2)_R$ Higgs-Yukawa model can be constructed using the Neuberger overlap operator $D^{(N)}$ (Lüscher).
- The model, we consider, is discretized on a four-dimensional lattice with $L$ sites per dimension (volume $V = L^4$).
- It contains one four-component, real Higgs field $\Phi$, and $N_f$ fermion doublets $\psi^{(i)}$, but no gauge fields:

$$Z = \int D\Phi \prod_{i=1}^{N_f} \left[ D\psi^{(i)} D\bar{\psi}^{(i)} \right] \exp \left( -S_{F}^{kin} - S_Y - S_\Phi \right)$$

with kinetic fermion action $S_{F}^{kin}$, Yukawa coupling term $S_Y$, and Higgs action $S_\Phi$. 
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with kinetic fermion action $S_F^{kin}$, Yukawa coupling term $S_Y$, and Higgs action $S_\Phi$.
- Kinetic fermion action:

$$S_F^{kin} = \sum_{i=1}^{N_f} \bar{\psi}^{(i)} D^{(N)} \psi^{(i)}$$
1.3 The model

- The Yukawa coupling term is given as

\[
S_Y = y_N \sum_{i=1}^{N_f} \bar{\psi}^{(i)} B \cdot \left[ 1 - \frac{1}{2\rho} D^{(N)} \right] \psi^{(i)}
\]

\[
B_{x,y} = 1_{x,y} \frac{(1 - \gamma_5)}{2} \phi_x + 1_{x,y} \frac{(1 + \gamma_5)}{2} \phi_x^\dagger
\]

where the Higgs field \( \Phi_x \) is written as quaternion \( \phi_x \) acting on flavor index

\[
\phi_x = \Phi_x^0 1 - i(\Phi_x^1 \tau_1 + \Phi_x^2 \tau_2 + \Phi_x^3 \tau_3), \quad \tau_i : \text{Pauli-matrices.}
\]
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\[ \phi_x = \Phi_x^0 \mathbb{1} - i(\Phi_x^1 \tau_1 + \Phi_x^2 \tau_2 + \Phi_x^3 \tau_3), \quad \tau_i : \text{Pauli-matrices}. \]

- The Higgs action \( S_\Phi \) in lattice notation is

\[ S_\Phi = -\kappa_N \sum_{x,\mu} \Phi_x^\dagger [\Phi_x + \mu + \Phi_{x-\mu}] + \sum_x \Phi_x^\dagger \Phi_x + \lambda_N \sum_x \left( \Phi_x^\dagger \Phi_x - N_f \right)^2 \]

related to usual notation by transforming couplings \((\kappa_N, \lambda_N) \leftrightarrow (\kappa, \lambda)\).
2.1 Phase structure at small $y_N$

- We consider the limit $N_f \to \infty$ where the couplings scale according to

$$y_N = \frac{\tilde{y}_N}{\sqrt{N_f}}, \quad \tilde{y}_N = \text{const} \quad \lambda_N = \frac{\tilde{\lambda}_N}{N_f}, \quad \tilde{\lambda}_N = \text{const} \quad \kappa_N = \tilde{\kappa}_N, \quad \tilde{\kappa}_N = \text{const}$$

- The effective action

$$S_{\text{eff}}[\Phi] = S_\Phi[\Phi] - N_f \cdot \log \left[ \det \left( y_N B \mathcal{D}^{(N)} - 2\rho \mathcal{D}^{(N)} - 2\rho B \right) \right]$$

can be evaluated at least for the constant and staggered modes of $\Phi$.

- For the Higgs field we take a magnetization ($m$) and a staggered magnetization ($s$) into account by the ansatz

$$\Phi(x) = \hat{\Phi} \cdot \sqrt{N_f} \cdot \left( m + s \cdot (-1)^{\sum_{\mu=0}^{3} x_\mu} \right)$$

where $\hat{\Phi} \in \mathbb{R}^4$, $|\hat{\Phi}| = 1$ is a constant unit vector and $m, s \in \mathbb{R}$.
2.2 Effective Potential

- At tree-level one finally finds for the effective potential \( V(m, s) \)

\[
\frac{V(m, s)}{L^4 N_f} = m^2 + s^2 - 8\tilde{\kappa}_N \left( m^2 - s^2 \right) + \tilde{\lambda}_N \left( m^4 + s^4 + 6m^2s^2 - 2 \left( m^2 + s^2 \right) \right) 
\]

\[
- \frac{1}{L^4} \sum_{p \in \mathcal{P}} \log \left[ \left| \nu^+(p) \right| \left| \nu^+(\varphi) \right| + \frac{\tilde{y}_N^2}{4\rho^2} \left( m^2 - s^2 \right) \left| \nu^+(p) - 2\rho \right| \left| \nu^+(\varphi) - 2\rho \right| \right]^2 
\]

\[
+ m^2 \frac{\tilde{y}_N^2}{4\rho^2} \left( \left| \nu^+(p) - 2\rho \right| \left| \nu^+(\varphi) \right| - \left| \nu^+(\varphi) - 2\rho \right| \cdot \left| \nu^+(p) \right| \right)^2 \right] \right]^2 
\]

with

\[
p = (p_0, p_1, p_2, p_3) \in \mathcal{P} : \text{allowed lattice momenta} \\
\nu(p) : \text{eigenvalues of Neuberger Dirac operator } \hat{D}^{(N)} \\
\varphi_\mu = p_\mu + \pi
\]

- The phase diagram can be explored by numerically searching for the absolute minima of the effective action with respect to \( m \) and \( s \).
2.3 Analytical phase diagrams

In general, four different phases can be obtained in this ansatz:

- **SYM**: \( m = 0, s = 0 \)
- **FM**: \( m \neq 0, s = 0 \)
- **AFM**: \( m = 0, s \neq 0 \)
- **FI**: \( m \neq 0, s \neq 0 \)
2.4 MC-simulations: Definitions

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- As observables we consider the (staggered) magnetization $m$ ($s$)

$$m = \left[ \sum_{i=0}^{3} \left| \frac{1}{L^4} \sum_{n} \Phi_i^n \right|^2 \right]^{\frac{1}{2}}, \quad s = \left[ \sum_{i=0}^{3} \left| \frac{1}{L^4} \sum_{n} (-1)^{n\mu} \cdot \Phi_i^n \right|^2 \right]^{\frac{1}{2}}$$

and the corresponding (staggered) susceptibility $\chi_m$ ($\chi_s$)

$$\chi_m = L^4 \cdot [\langle m^2 \rangle - \langle m \rangle^2], \quad \chi_s = L^4 \cdot [\langle s^2 \rangle - \langle s \rangle^2],$$

where $\langle \ldots \rangle$ denotes the average over the generated $\Phi$-field configurations.
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\]

where \( \langle \ldots \rangle \) denotes the average over the generated \( \Phi \)-field configurations.
- Determine phase transition point by fit of \( \chi_{m,s} \) to finite-size-scaling ansatz

\[
\chi_{m,s} = A_{1}^{m,s} \cdot \left( \frac{1}{L^{-2/\nu} + A_{2,3}^{m,s} (\kappa N - \kappa_{\text{crit}})^2} \right)^{\gamma/2},
\]

with fitting parameters \( \kappa_{\text{crit}}^{m,s}, A_{1}^{m,s}, A_{2}^{m,s}, A_{3}^{m,s} \).
2.5 Phase structure overview

- Numerically we find the expected phases at the predicted locations.
- Qualitatively, the phase diagram is in very good agreement with the large $N_f$ analysis.

\[
\tilde{\lambda}_N = 0.1, N_f = 10
\]

![Phase structure diagram](image)
2.6 Finite Size Effects

- Phase transition lines strongly shifted by finite size effects.
- We isolate finite size effects from $1/N_f$-corrections by choice $N_f = 50$.
- We compare $L = 4$ and $L = 8$ results with analytical finite size expectations.
2.7 $1/N_f$ corrections

- We demonstrate strength of $1/N_f$-corrections by determining phase transition points $\kappa_{\text{crit}}^{m,s}$ for several values of $N_f$.
- To isolate $1/N_f$-corrections from finite size effects we compare with analytical, finite size expectations.

\begin{align*}
\tilde{y}_N &= 1.0, \tilde{\lambda}_N = 0.1 \\
\tilde{y}_N &= 2.0, \tilde{\lambda}_N = 0.1
\end{align*}
3.1 Phase structure at large $y_N$

- Idea: Divide out $y_N B(D^{(N)} - 2\rho)$ and develop logarithm into power series

$$S_{eff}[\Phi] = S_\Phi - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2\rho D^{(N)} - 2\rho B \right) \right]$$
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\[
\rightarrow S_{\Phi} - N_f \cdot \log \left[ \det \left( 1 - \frac{2\rho}{y_N} D^{(N)} \left[ D^{(N)} - 2\rho \right]^{-1} B^{-1} \right) \right]
\]
\[
\rightarrow S_{\Phi} - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}
\]

where the coupling matrix $K_{x,y}$ is explicitly known.
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\]

\[
\rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \left( \frac{4\rho}{y_N^2} \right)^2 \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}
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where the coupling matrix $K_{x,y}$ is explicitly known.

- In large $N_f$-limit amplitude $|\Phi_x|$ becomes fixed.

The model becomes an $O(4)$-symmetric sigma-model leading to the existence of a symmetric phase at strong Yukawa couplings.
3.1 Phase structure at large $y_N$

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$$S_{eff}[\Phi] = S_\Phi - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2\rho D^{(N)} - 2\rho B \right) \right]$$

$$\rightarrow S_\Phi - N_f \cdot \log \left[ \det \left( \mathbb{1} - \frac{2\rho}{y_N} D^{(N)} \left[ D^{(N)} - 2\rho \right]^{-1} B^{-1} \right) \right]$$

$$\rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}$$

where the coupling matrix $K_{x,y}$ is explicitly known.

- **Caution:** $D^{(N)} - 2\rho$ has zero modes: More careful calculation yields

$$S_{eff}[\Phi] \rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}$$

$$- N_f \cdot \log \det^* \left( B^{-1} \right) - N_f \cdot \log \det^* \left( \mathbb{1} + \frac{2\rho}{y_N} F[\Phi] \right)$$

where $\det^*$ is the determinant over zero-modes (120 modes) and $F[\Phi]$ can be explicitly given.
3.2 Analytical Phase Diagrams

- Neglecting the finite-volume terms, the phase structure can be derived in the large $N_f$-limit applying the ansatz

$$y_N = \tilde{y}_N, \quad \tilde{y}_N = \text{const}, \quad \lambda_N = \frac{\tilde{\lambda}_N}{N_f}, \quad \tilde{\lambda}_N = \text{const}, \quad \kappa_N = \frac{\tilde{\kappa}_N}{N_f}, \quad \tilde{\kappa}_N = \text{const},$$
3.3 MC-results: Magnetizations

- We show the (staggered) magnetization for varying $\kappa_N$ at $\tilde{\lambda}_N = 0.1$, $\tilde{y}_N = 30$ for different lattice sizes.

- Strong finite volume effects prevent emergence of the symmetric phase on too small lattices and cause asymmetry in $m$ and $s$. 

![Graphs showing magnetizations for different lattice sizes](image-url)
3.3 MC-results: Magnetizations

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\begin{align*}
V = 4^4 & \quad V = 8^4 & \quad V = 16^4
\end{align*}

- Strongest finite-volume contribution is $\log \det^* (B^{-1})$.
  It can be written in terms of $m$ and $s$, explaining the observations

$$
-N_f \log \det^* (B^{-1}) = \text{Const} - 8N_f \log |m| + 64N_f \log |m^2 - s^2|
$$
3.4 MC-results: Susceptibilities

- We show the magnetic susceptibilities for varying $\kappa_N$ at $\tilde{\lambda}_N = 0.1$, $\tilde{y}_N = 30$ for different lattice sizes.
- On small lattices ($V = 4^4$) the maximum is at $\kappa_N = 0$, caused by the finite-volume effects.
- On larger lattices ($V = 8^4$) a second peak develops at $\kappa_N = 0.04$, which describes the location of the physical phase transition.
3.5 Phase Diagram

- We compare numerical and analytical results for the SYM-FM transition line.
- Good agreement is found even at $N_f = 2$.
- The SYM-AFM phase transition line was numerically too demanding for our HMC-algorithm.

![Phase Diagram](image)
4.1 Towards Upper Mass Bounds:

- VERY PRELIMINARY
- To access physical setting $N_f = 1$, we implemented a PHMC-algorithm.
- We fixed physical scale by phenomenological value $vev = 246$ GeV.
- We searched for the physical region in the phase diagram by fixing the top quark mass to $m_{top} = 175$ GeV.
- To obtain upper mass bound, we went to strong quartic couplings $\lambda_N$.
- As a first step, we simulated the model on a $16^3 \times 32$ lattice close to the phase transition in the FM-phase.
- To account for the 3 Goldstone-modes we split the 4-component Higgs field into its radial $\phi$ and tangential components $\vec{\pi}$. 
4.2 Goldstone - Propagator:

- Obtain Goldstone renormalization factor $Z_G$ from inverse propagator of massless Goldstone-modes

\[ G^{-1}_\pi (\hat{p}^2) = \frac{\hat{p}^2}{Z_G} \]

\[ Z_G = 0.9683 \pm 0.0002 \]
4.3 Higgs - Propagator:

- Obtain Higgs propagator-mass $m_{H,prop}$ from propagator of Higgs-mode

$$G^{-1}_\phi (\hat{p}^2) = \frac{\hat{p}^2 + m^2_{H,prop}}{Z_H}$$

$m_{H,prop} = 0.384 \pm 0.009$
4.4 Fermion correlator:

- Obtain top quark mass $m_{top}$ from fermion correlator $\langle \psi_{t_1} \bar{\psi}_{t_2} \rangle$

$$m_{top} = 0.0686 \pm 0.0021$$
4.5 Higgs correlator:

- Obtain Higgs mass $m_H$ from Higgs correlator $\langle \phi_{t_1} \phi_{t_2} \rangle$

\[
\langle \phi_{t_1} \phi_{t_2} \rangle
\]

\[
\Delta t = |t_2 - t_1|
\]

\[
m_H = 0.286 \pm 0.011
\]
4.5 Summary of results:

Cut-off $\Lambda$ \hspace{2cm} $(2591 \pm 58)$ GeV
Top mass $m_{top}$ \hspace{2cm} $(178.8 \pm 6.8)$ GeV
Higgs mass $m_H$ \hspace{2cm} $(741 \pm 29)$ GeV
Higgs prop. mass $m_{H,prop}$ \hspace{2cm} $(994 \pm 22)$ GeV

Bare Lambda $\lambda_0$ \hspace{2cm} 4.4
Ren. Lambda $\lambda_{ren}$ \hspace{2cm} $4.53 \pm 0.18$
Ren. $y$ $y_{ren}$ \hspace{2cm} $0.723 \pm 0.027$
Summary and Outlook

- Large $N_f$ analysis gives good understanding of qualitative phase structure.
- Finite size effects can be quantitatively described in large $N_f$-limit.
- A symmetric phase exists also at strong Yukawa coupling.
- First results on upper Higgs mass bound will become available soon.
- We will investigate the model at larger cut-offs and on larger lattices.
Evidence for FI-phase

• Also numerical evidence for the predicted FI-phase with

\[ \langle m \rangle > 0 \text{ and } \langle s \rangle > 0 \]

depth inside the anti-ferromagnetic phase can be found.

• The plots were made for \( \tilde{\lambda}_N = 0.1, N_f = 10, L = 6 \).
Finite Size Effects

Magnetizations

Susceptibility $\chi_m$

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