The Yang-Mills Vacuum in 2+1 Dimensions

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From time to time, people have tried to obtain the Yang-Mills vacuum wavefunctional, often in lower dimensions, to see if anything can be learned about confinement and the mass spectrum.

The idea is to solve for the ground state, e.g. in temporal gauge

\[
H \Psi_0 = E_0 \Psi_0
\]

to find out if spacelike Wilson loops have an area law falloff

\[
W(C) = \langle \Psi_0 | \text{Tr} \exp[i \oint_C A] | \Psi_0 \rangle \sim \exp[-\sigma \text{Area}(C)]
\]

and if so, why.
Our claim is that the ground state solution in D=2+1 dimensions is approximated by

\[
\Psi_0[A] = \exp \left[ -\frac{1}{2} \int d^2x d^2y \ B^a(x) \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} \ B^b(y) \right]
\]

where

\[ B^a = F^a_{12} \]

is the color magnetic field strength

\[ D^2 = D_k D_k \]

is the covariant Laplacian in adjoint color representation,

\[ \lambda_0 \]

is the lowest eigenvalue of \(-D^2\)

\[ m \]

is a constant proportional to \(g^2\)

Previous relevant work by:

J.G. (1979)
Samuel (1996)
Diakonov (unpublished)
In support of this claim, we argue that this expression for $\Psi_0$

- is a solution of the YM Schrodinger equation in the $g \to 0$ limit;

- solves the YM Schrodinger equation in the strong field, zero-mode limit;

- confines if $m > 0$, and that $m > 0$ seems energetically preferred;

- results in the numerically correct relationship between the mass gap and the string tension.
To begin at the beginning:

In Yang-Mills theory quantized in temporal gauge, all physical states must satisfy the Gauss Law constraint

\[
\left( \delta^{ac} \partial_k + g \epsilon^{abc} A^b_k \right) \frac{\delta}{\delta A^c_k} \Psi = 0
\]

which is equivalent to invariance of \( \Psi[A] \) under infinitesimal gauge transformations. The Hamiltonian is

\[
H = \int d^d x \left\{ -\frac{1}{2} \frac{\delta^2}{\delta A^a_k(x)^2} + \frac{1}{4} F^a_{ij}(x)^2 \right\}
\]
**Free Field Limit**

The proposed ground state

\[
\Psi_0[A] = \exp \left[ -\frac{1}{2} \int d^2xd^2y \ B^a(x) \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)^{ab}_{xy} \ B^b(y) \right]
\]

obviously satisfies the non-abelian physical state condition, and in the \( g \to 0 \) limit this becomes

\[
\Psi_0[A] = \exp \left[ -\int d^2xd^2y \left( \partial_1 A_2^a(x) - \partial_2 A_1^a(x) \right) \right.
\]
\[
\times \left( \frac{\delta^{ab}}{\sqrt{-\nabla^2}} \right)_{xy} \left( \partial_1 A_2^b(y) - \partial_2 A_1^b(y) \right) \right]
\]

which is the known ground state solution in the abelian, free-field case.
Zero Mode Limit

Consider gauge fields constant in space, variable in time, in D=2+1 dimensions. Lagrangian

\[ L = \frac{1}{2} \int d^2 x \left[ \partial_t A_k \cdot \partial_t A_k - g^2 (A_1 \times A_2) \cdot (A_1 \times A_2) \right] \]

Hamiltonian operator

\[ H = -\frac{1}{2} \frac{1}{V} \frac{\partial^2}{\partial A^a_k \partial A^a_k} + \frac{1}{2} g^2 V (A_1 \times A_2) \cdot (A_1 \times A_2) \]

Vacuum state

\[ \Psi_0 = \exp[-V R_0] \]
With some algebra, one can verify that

\[ \Psi_0 = \exp \left[ -\frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{|A_1|^2 + |A_2|^2}} \right] \]

solves the zero-mode YM Schrodinger equation up to \( 1/V \) corrections.

Then we consider our proposal for the full vacuum state, for vacuum fluctuations in the \textbf{strong A-field limit}, where the covariant Laplacian is dominated by the gauge-field zero-mode, i.e.

\[ (-D^2)_{xy}^{ab} = g^2 \delta(x - y) \left[ (A_1^2 + A_2^2) \delta^{ab} - A_1^a A_1^b - A_2^a A_2^b \right] \]
Then one finds (using also that $B \perp A_1, A_2$ in SU(2) color space)

\[ \Psi_0[A] = \exp \left[ - \int d^2x d^2y \, B^a(x) \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)^{ab}_{\,xy} B^b(y) \right] \]

\[ \implies \exp \left[ - \frac{1}{2} g V \left( A_1 \times A_2 \right) \cdot \left( A_1 \times A_2 \right) \right] \]

So our wavefunctional

- satisfies the physical state constraint;
- has the proper perturbative $g \to 0$ limit.
- agrees with the calculable ground state of the zero-mode limit.

**Supposing its right, what about confinement?**
Dimensional Reduction

A long time ago it was suggested that at large distance scales, the pure Yang-Mills vacuum in a confining theory looks like

\[ \Psi_{0}^{\text{eff}} \approx \exp \left[ -\mu \int d^{d}x \ F_{ij}^{a}(x)F_{ij}^{a}(x) \right] \]

This vacuum state has the property of \textit{dimensional reduction}: Computation of a spacelike loop in \( d+1 \) dimensions reduces to the calculation of a Wilson loop in Yang-Mills theory in \( d \) Euclidean dimensions.

J.G. (1979)
Mode number cutoff: Expand $B(x)$ in eigenmodes of the covariant Laplacian:

\[
(-D^2)^{ab} \phi^b_n(x) = \lambda_n \phi^a_n(x)
\]

\[
B^a(x) = \sum_{n=0}^{\infty} b_n \phi^a_n(x)
\]

\[
B^{a,\text{slow}}(x) = \sum_{n=0}^{n_{\text{max}}} b_n \phi^a_n(x)
\]

The cutoff mode sum defines the “slowly varying” $B$-field. Choosing $n_{\text{max}}$ such that $\lambda_{n_{\text{max}}} - \lambda_0 \ll m^2$

\[
\int d^2 x d^2 y \ B^{a,\text{slow}}(x) \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^{b,\text{slow}}(y) \approx \frac{1}{m} \int d^2 x \ B^{a,\text{slow}}(x) B^{a,\text{slow}}(x)
\]
So the part of the squared wavefunctional that involves $B_{\text{slow}}$ is

$$|\Psi_0|^2 = \exp \left[ -\frac{1}{m} \int d^2x \, B_{\text{slow}} B_{\text{slow}} \right]$$

which is the probability distribution of D=2 dimensional Yang-Mills (i.e. dimensional reduction). The string tension $\sigma$ can be calculated analytically; in lattice units it is

$$\sigma = \frac{3}{4} \frac{m}{\beta}$$

Suppose we turn this around, and fix $m = \frac{4}{3} \beta \sigma$, with $\sigma$ taken from the Monte Carlo data. Then the full vacuum wavefunctional

$$\Psi_0[A] = \exp \left[ -\frac{1}{2} \int d^2x d^2y \, B^a(x) \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^b(y) \right]$$

must imply a definite value for the mass gap. What is it?
**Numerical Simulation of** $|\Psi_0|^2$

To get the mass gap, we need to compute the connected correlator

$$G(x - y) = \langle (B^a B^a)_x (B^b B^b)_y \rangle - \langle (B^a B^a)_x \rangle^2$$

in the probability distribution

$$P[A] = |\Psi_0[A]|^2 = \exp \left[ - \int d^2 x d^2 y \ B^a(x) K_{xy}[A] B^b(y) \right]$$

where

$$K_{xy}[A] = \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)^{ab}_{xy}$$

**Numerically, this looks hopeless!** $K_{xy}$ is highly non-local, and is not even known explicitly for arbitrary gauge fields.

We have, nonetheless, found a way to do the simulation....
Observables of interest include

- The eigenvalue spectrum \( \{\lambda_n\} \) of the adjoint covariant Laplacian \((-D^2)\)

- The connected field-strength correlator

\[
\langle B^2(x)B^2(y) \rangle_{\text{conn}} \propto G(x - y)
\]

where

\[
G(x - y) = \left\langle (K^{-1})_{xy} (K^{-1})_{yx} \right\rangle
\]

\[
K^{-1} = \sqrt{-D^2 - \lambda_0 + m^2}
\]

with the parameter \( m \) chosen to reproduce the known string tension \( \sigma \)

\[
m = \frac{4}{3} \beta \sigma
\]

From \( G(R) \), we can extract the mass gap.
For Comparison

We can also compute \( \{\lambda_n\}, \ K_{xy}^{ab}, \) and

\[
G(x - y) = \left\langle (K^{-1})_{xy}^{ab} (K^{-1})_{yx}^{ba} \right\rangle
\]

\[
K^{-1} = \sqrt{-D^2 - \lambda_0 + m^2}
\]

on 2D slices of lattices generated by 3D lattice Monte Carlo.

This is like simulating the ground state of the transfer matrix in the Euclidean theory.

Results obtained from “MC” lattices, generated by ordinary lattice Monte Carlo, can be compared with results obtained by simulating \( |\Psi_0|^2 \) (“recursion” lattices).
**Eigenvalue Spectrum**  \( \beta=18, \text{ 50x50 lattice} \)

This is a plot of eigenvalue vs mode number of

- the zero-field operator \((-\nabla^2 + m^2)\)
- the covariant operator \((-D^2 - \lambda_0 + m^2)\), computed on 10 lattices.

These are *not* averaged; the values for each lattice are plotted, and (almost) fall on top of one another.

There is **very little variance** in the spectrum of \(-D^2-\lambda_0\) from one lattice to the next.
Mass Gap

Here is the data for

\[ G(x - y) = \left\langle \left( K^{-1} \right)_{x y}^{a b} \left( K^{-1} \right)_{y x}^{b a} \right\rangle \]

The data is obtained from ten recursion lattices, and ten MC lattices. Note the tiny values of \( G(R) \) obtained at larger \( R \). This requires a near-absence of fluctuation in \( K^{-1} \) from one lattice to the next.
The mass gap is obtained by fitting the data for $G(R)$ to extract the exponential falloff. Define

$$G_0(R) = \delta^{ab} \delta^{ba} \left[ \left( \sqrt{-\nabla^2 + \mu^2} \right)_{xy} \right]^2$$

$$= \frac{3}{4\pi^2} \left( 1 + \frac{1}{2} MR \right)^2 \frac{e^{-MR}}{R^6}$$

where $R = |x-y|$ and $M = 2\mu$. 

![Graph showing correlation fit, $\beta=18$, $L=50$]
Results for the mass gap

- “recursion” is our result.
- “expt” is the Monte Carlo result for the $0^+$ glueball, obtained by Meyer and Teper.

Given string tension $\sigma$, we have determined fairly accurately the $0^+$ glueball mass.
**Why Confinement?**

Our wavefunction confines if \( m \neq 0 \). So “why confinement” in 2+1 boils down to “why is \( m \neq 0 \)?”

An obvious approach is to treat \( m \) as a variational parameter, and use it to minimize \( \langle H \rangle \). To simplify matters, noting the comparative lack of fluctuation in

\[
K = \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}
\]

we

- neglect functional derivatives of the kernel \( K \) in computing \( \langle H \rangle \); and
- ignore correlations between \( B \) and \( K \).
Then we find

\[
\langle H \rangle = \frac{1}{2} \left\langle \text{Tr} \sqrt{-D^2 - \lambda_0 + m^2} + \frac{1}{2} \text{Tr} \frac{\lambda_0 - m^2}{\sqrt{-D^2 - \lambda_0 + m^2}} \right\rangle
\]

increase with \( m \) \hspace{1cm} \text{decrease with} \ m

In an abelian theory where

\[
D^2 \rightarrow \nabla^2
\]

\[
\lambda_0 = 0
\]

the first term wins, and \( \langle H \rangle \) is minimized at \( m=0 \), as it should be.
The situation is different in the non-abelian theory. We compute

\[
\langle H \rangle = \frac{1}{2} \left\langle \text{Tr} \sqrt{-D^2 - \lambda_0 + m^2} + \frac{1}{2} \text{Tr} \frac{\lambda_0 - m^2}{\sqrt{-D^2 - \lambda_0 + m^2}} \right\rangle
\]

by simulation of \( |\Psi_0|^2 \). Result for \( \beta=6 \):

Note the tiny variation of vacuum energy density with \( m \!\!).

Minimum around \( m=0.3 \).

To get the right string tension, we would need \( m=0.515 \).
What about N-ality?

$\Psi_0$ has the dimensional reduction form at large scales

$$
\Psi_0[A] = \exp \left[ -\frac{1}{2m} \int d^2x \ B^2 \right]
$$

2D Yang-Mills --> Casimir scaling.

However, Casimir scaling is only true at intermediate distances (or at $N=\infty$). At large enough scales, the string tension should depend only on the N-ality of the static color charges, due to color screening by gluons.

So - what about color screening?
How is this problem solved at strong couplings, where we can solve for the ground state analytically? \textit{(JG, 1980)}

The vacuum is

\[ \Psi_0[U] = \exp[R(U)] \]

where, up to $O(\beta^4)$

\[ R[U] = \sum \text{contours} \quad c_0 \quad + \quad c_1 \quad + \quad c_2 \quad + \quad c_3 \quad + \quad \text{larger contours} \]
The 1X2 rectangle screens adjoint loops.

It also gives the leading correction to dimensional reduction.

Expansion in powers of lattice spacing:

$$\Psi_0[U] = \exp\left[-\frac{2}{\beta} \int d^2 x \left( a\kappa_0 B^2 - a^3\kappa_2 B(-D^2)B + \ldots \right)\right]$$

where (Guo et al, 94)

\[
\kappa_0 = \frac{1}{2} c_0 + 2(c_1 + c_2 + c_3)
\]

\[
\kappa_2 = \frac{1}{4} c_1
\]

Note! Leading correction comes from the rectangle \(\propto c_1\)
That's strong coupling. Returning to our proposal, expand the kernel in powers of $1/m^2$

$$\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} = \frac{1}{m} \left( 1 - \frac{-D^2 - \lambda_0}{2m^2} + \ldots \right)$$

Then the part of the vacuum that depends on $B^{\text{slow}}$ is

$$\exp \left[ -\frac{1}{m} \int d^2x \left( B^{\text{slow}} B^{\text{slow}} - B^{\text{slow}} \frac{-D^2 - \lambda_0}{2m^2} B^{\text{slow}} + \ldots \right) \right]$$

Leading correction is similar (and has the same sign) to the leading correction to diml reduction produced by the rectangle term.

*Maybe this correction is responsible for color screening?*

This term allows $B$ (gluons) to “propagate”, and, perhaps, break adjoint strings.
Conclusions

We have presented a proposal for the YM vacuum state in D=2+1 which

- is a solution of the YM Schrodinger equation in the $g \to 0$ limit;

- solves the YM Schrodinger equation in the strong field, zero-mode limit;

- confines if $m > 0$, and $m > 0$ seems energetically preferred

- results in the numerically correct relationship between the string tension and mass gap.
Open questions:

- do first corrections to dimensional reduction really give the right N-ality dependence

- can we improve the variational calculation?

- extension to D=3+1 ??
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in the probability distribution

$$P[A] = |\Psi_0[A]|^2 = \exp \left[ - \int d^2 x d^2 y \ B^a(x) K^{ab}_{xy}[A] B^b(y) \right]$$

where

$$K^{ab}_{xy}[A] = \left( \frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)^{ab}_{xy}$$

**Numerically, this looks hopeless!** $K^{ab}_{xy}$ is highly non-local, and is not even known explicitly for arbitrary gauge fields.

We have, nonetheless, found a way to do the simulation....
General idea of the simulation: work in an axial $A_1 = 0$ gauge, and change integration variables from $A_2$ to $B$. Then:

1. given $A_2$, set $A'_2 = A_2$

2. $P[A; K[A']]$ is gaussian in $B$. Diagonalize $K_{ab}^{xy}[A']$ and generate a new $B$-field (or set of $B$-fields) stochastically.

3. from $B$, calculate $A_2$ in axial gauge, and compute observables

4. go to step 1, repeat as necessary.

(all in a lattice regularization)

\[
P^{(1)}[A] = P[A; K[0]]
\]

\[
P^{(n+1)}[A] = \int DA' P[A; K[A']] P^{(n)}[A']
\]
But suppose - after eliminating the variance along gauge orbits by a gauge choice - that $K[A]$ has very little variation among thermalized configurations. Then things are more promising.

Define

$$P[A; K[A']] = \exp \left[ - \int d^2x d^2y \ B^a(x) K_{xy}^{ab}[A'][B^b(y)] \right]$$

where $B$ is computed from $A$, not $A'$, and $P[A] = P[A, K[A]]$. Then, assuming the variance of $K$ is small,

$$P[A] \approx P[A, \langle K \rangle]$$

$$= P \left[ A, \int DA' \ K[A'] P[A'] \right]$$

$$\approx \int DA' \ P \left[ A, K[A'] \right] P[A']$$

solve this equation iteratively...
Observables of interest include

- The eigenvalue spectrum \( \{\lambda_n\} \) of the adjoint covariant Laplacian \((-D^2)\)

- The connected field-strength correlator

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where

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with the parameter \( m \) chosen to reproduce the known string tension \( \sigma \)

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