Complete spectra of the staggered Dirac operator and their relation to Polyakov loops

Christian Hagen

In collaboration with:
E. Bilgici, F. Bruckmann and C. Gattringer

Universität Regensburg, Universität Graz

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Outline

1. Definitions
2. Spectral sums
3. Simulation details
4. Results
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Definitions

- Polyakov loop averaged over space:

\[
P = \frac{1}{V_3} \sum_{\vec{n}} L(\vec{n}) \quad \text{with} \quad L(\vec{n}) = Tr_c \left[ \prod_{n_4=1}^{N_t} U_4(\vec{n}, n_4) \right]
\]

- Staggered Dirac operator at \(am_q = 0\):

\[
D(n, m) = \frac{1}{2} \sum_{\mu=1}^{4} \eta_{\mu}(n) \left[ U_{\mu}(n) \delta_{n+\hat{\mu}, m} - U_{\mu}(n - \hat{\mu})^\dagger \delta_{n-\hat{\mu}, m} \right]
\]

with \(\eta_{\mu}(n) = (-1)^{n_1 + \cdots + n_{\mu-1}}\).
Powers of $D$ \rightarrow \text{products of links} \rightarrow \text{loops} \ (\text{PRL} \ 97 \ (2006) \ 032003)

\[ Tr_c D^{N_t}(n, n) = \frac{1}{2^{N_t}} Tr_c \prod_{s=1}^{N_t} U_4(\vec{n}, s) - \frac{1}{2^{N_t}} Tr_c \prod_{s=0}^{N_t-1} U_4(\vec{n}, N_t - s) + \text{other loops, trivially closed} \]

\[ = \frac{1}{2^{N_t}} [L(\vec{n}) - L^*(\vec{n})] + \text{other loops, trivially closed} \]
Changing temporal boundary conditions

\[ U_4(\vec{n}, N_t) \rightarrow z U_4(\vec{n}, N_t), \quad |z| = 1, \]

only affects the Polyakov loops

\[ Tr_c D_{z}^{N_t}(n, n) = \frac{1}{2^{N_t}} \left[ zL(\vec{n}) - z^*L^*(\vec{n}) \right] + \text{other loops}. \]

Averaging over space and time gives

\[ Tr \ D_{z}^{N_t} = \frac{V_4}{2^{N_t}} (zP - z^*P^* + X), \]

where \( X \) is the sum of all trivially closed paths.
Spectral sums

Generalization to arbitrary one-link operators

\[ \text{Tr}(D_{z_t}^N) = C (zP \pm z^* P^* + X), \]

where \( C \) is some trivial constant depending on the operator.

Linear combination for three different boundary conditions:

\[
\frac{1}{C} (a_1 \text{Tr}(D_{z_1}^N) + a_2 \text{Tr}(D_{z_2}^N) + a_3 \text{Tr}(D_{z_3}^N)) = \\
= P \left( z_1 a_1 + z_2 a_2 + z_3 a_3 \right) \pm P^* \left( z_1^* a_1 + z_2^* a_2 + z_3^* a_3 \right) \\
\overset{\parallel 1}{=} 1 \\
+ X (a_1 + a_2 + a_3) \overset{\parallel 0}{=} P
\]
Spectral sums

Polyakov loop averaged over space:

- from the links

Definition:

\[
P = \frac{1}{V_3} \sum_{\vec{n}} \text{Tr}_c \left[ \prod_{n_4=1}^{N_t} U_4(\vec{n}, n_4) \right]
\]

- from the eigenvalues \( \lambda \) of the staggered Dirac operator, e.g. for \( z_1 = 1 \) and \( z_2 = z_3^* = e^{i\frac{2\pi}{3}} \equiv z \)

\[
P = \frac{2^{N_t}}{V_4} \left[ \sum_i (\lambda^{(i)})^{N_t} + z^* \sum_i (\lambda_z^{(i)})^{N_t} + z \sum_i (\lambda_{z^*}^{(i)})^{N_t} \right]
\]
## Simulation details

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<tr>
<th>$\beta$</th>
<th>$L^3 \times N_t$</th>
<th>$a[fm]$</th>
<th>$T[MeV]$</th>
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<td>$6^3 \times 4, 12^3 \times 4$</td>
<td>0.351(3)</td>
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<td>0.146(2)</td>
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<td>8.06</td>
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<td>0.129(1)</td>
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<td>8.40</td>
<td>$12^3 \times 6$</td>
<td>0.098(1)</td>
<td>337</td>
</tr>
</tbody>
</table>

- Lüscher-Weisz gauge action
- $z_1 = 1$ and $z_2 = z_3^* = e^{i\frac{2\pi}{3}} \equiv z$
- Eigenvalues are ordered w.r.t. their absolute values
Results

Distribution of the eigenvalues

- $6^3 \times 4$
- $12^3 \times 4$
- $12^3 \times 6$
Averaged shift of the eigenvalues

\[ s(\lambda^{(i)}) = \left( |\lambda^{(i)} - \lambda_z^{(i)}| + |\lambda^{(i)} - \lambda_z^{(i)*}| + |\lambda_z^{(i)} - \lambda_z^{(i)*}| \right) / 3 \]
Results
Contribution of individual eigenvalues to the Polyakov loop

\[ c(\lambda^{(i)}) = \frac{2^{N_t}}{V^4} \left[ (\lambda^{(i)})^{N_t} + z^* (\lambda_z^{(i)})^{N_t} + z (\lambda_{z^*}^{(i)})^{N_t} \right] \text{ with } P = \sum_i c(\lambda^{(i)}) \]
Results

Accumulated contributions of the eigenvalues to the Polyakov loop

![Graphs showing accumulated contributions of eigenvalues for different temperatures and lattice sizes.](image)
Results

Phase shift of the truncated spectral sums for $T > T_c$

**$12^3 \times 4$**

**$12^3 \times 6$**
Results

Explanation of phase shift

- Eigenvalues of staggered Dirac Operator are purely imaginary $\Rightarrow \lambda^{4n}$ is real, positive number and $\lambda^{4n+2}$ is real, negative number.

- First term in truncated sum:

$$c(\lambda^{(1)}) = \frac{2^{N_t}}{V_4} \left[ \left( \lambda^{(1)} \right)^{N_t} + z^* \left( \lambda_z^{(1)} \right)^{N_t} + z \left( \lambda_z^{(1)} \right)^{N_t} \right]$$

$$= \frac{2^{N_t}}{V_4} \left[ \alpha + z^* \tilde{\alpha}_z + z \tilde{\alpha}_z^* \right]$$

$$\tilde{\alpha}_z^* \approx \tilde{\alpha}_z \equiv \tilde{\alpha} \overset{\ddagger}{=\frac{2^{N_t}}{V_4} \left[ \alpha + (z^* + z)\tilde{\alpha} \right] = \frac{2^{N_t}}{V_4} \left[ \alpha - \tilde{\alpha} \right]} {< 0}$$

$$\overset{\ddagger}{=\frac{2^{N_t}}{V_4} \left[ \alpha + (z^* + z)\tilde{\alpha} \right] = \frac{2^{N_t}}{V_4} \left[ \alpha - \tilde{\alpha} \right]} {> 0}$$

Synatschke, Wipf, Wozar, hep-lat/0703018

T. Kovacs, private communication
Summary:

- IR modes are shifted most
- Mainly eigenvalues in the UV build up the Polyakov loop
- Phase shift of $180^\circ$ for the accumulated contribution for $N_t = 4n$ in the IR but also for $N_t = 4n + 2$, here in between
- IR modes know about QCD phase transition

Outlook:

- Closer investigation on how the eigenvalues respond to change of b.c.
- Investigation of other one-link operators, e.g., Laplace operator
- Derivation of eigenvalue formula with well-defined continuum limit