Surprises with the lattice index theorem

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Center Vortices and Topology
Atiah-Singer Index Theorem
Methods to determine the Topological charge $Q$:

- Cooling for rough configurations...
- Without cooling for smooth configurations from $F\tilde{F}$
- Topology of Center Vortices
- From Zero-modes via Index Theorem
Center Vortices

P-Vortices: closed surfaces of quantised flux

\[ d^2\sigma_{\mu\nu} = \epsilon_{ab} \frac{\partial \bar{x}_\mu}{\partial \sigma_a} \frac{\partial \bar{x}_\nu}{\partial \sigma_b} d^2\sigma \]

\[ Q = \text{Topological winding number} \]
\[ Q = \text{Self intersection number} \]

\[ \rightarrow \text{Engelhardt, Reinhardt (2000)} \]

\[ Q = -\frac{1}{16} \epsilon_{\mu\nu\alpha\beta} \int_S d^2\sigma_{\alpha\beta} \int_S d^2\sigma'_{\mu\nu} \delta^4(\bar{x}(\sigma) - \bar{x}(\sigma')) \]

one intersection contributes \( \pm \frac{1}{2} \)

Specify surface orientation!
Contributions to topological charge $Q$

vortex intersection

$4 \cdot 4 = 16$ contributions

$Q = \pm \frac{1}{2}$

1 contribution

$Q = \pm \frac{1}{32}$
contributions to topological charge

- intersections
- writhing points
Exact zero-modes and the Atiyah-Singer index theorem

- Topological charge:

\[ Q := -\frac{N_f}{16\pi^2} \int d^4x \text{tr} (F_{\mu\nu} \tilde{F}_{\mu\nu}) \]

- Index theorem:

\[ n_-, n_+ : \text{number of left-/right-handed zeromodes} \]

\[ \text{ind } D[A] = n_- - n_+ = Q[A] \]
Localisation

Scalar density

\[ \rho(x) = \sum_{c,d} |\vec{v}(x)_{cd}|^2, \]

c and d, color and Dirac indices

Chiral densities \( \rho_+(x) \) and \( \rho_-(x) \)

\[ \rho_{\pm}(x) = \sum_{c,d} \bar{v}(x)^*_{cd} \frac{1 - \gamma^c_{5,d'}}{2} v(x)_{cd'}^2 \]
Analytical results

⇒ Reinhardt, Schroeder, Tok and Zhukovsky (2002)

Analytical calculations by the Tübingen group. Zero-modes peak at intersections of vortices.

Probability density of zero-mode in the background of four intersecting vortices.
Plane Vortices

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Topology

Center

Vortices

Index theorem

Analytical results

Plane Vortices

Spherical Vortices

Conclusions

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Geometry

Topological charge

Fermionic density

Index of the overlap operator: \( \text{ind } D \equiv 0 \)

Topological charge: \( Q = 0 \)

Conclusion: index theorem valid
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Topology Center
Vortices

Index theorem
Analytical results
Plane Vortices
Spherical Vortices
Conclusions

Thick Spherical SU(2)-vortices, \(40^3 \times 2\)-lattice

\[
U_{\mu}(x^\nu) = \begin{cases} 
\exp\{i\alpha(r)\frac{\vec{r}}{r}\sigma\}, & t = 1, \mu = 4 \\
1 & \text{else} 
\end{cases}
\]

\[
L(\vec{r}) = \exp\{i\alpha(r)\frac{\vec{r}}{r}\vec{T}\} \quad \text{time-like Wilson lines}
\]

\[
1 - \frac{1}{2} \text{tr } U_\Box \leq 0.015 
\]
Non-/Orientable Spherical Vortices

Orientation of the vortex surface assigned by abelian projection. Non-orientable vortex surface leads to monopole lines.
Orientable Spherical Vortices

- No intersection/writhing points
- Index of the overlap operator: \( \text{ind } D = 0 \)
- Topological charge before cooling: \( Q = 0 \)
- Topological charge after cooling: \( Q = 0 \)
Non-orientable Spherical Vortices

- No intersection/writhing points
- Index of the overlap operator: \( \text{ind } D = 1 \)
- Topological charge before cooling: \( \vec{E} \neq 0, \vec{B} = 0 \rightarrow Q = 0 \)
- Topological charge after cooling: \( Q = 1 \)
Non-orientable Spherical Vortex

- Index of overlap operator:

\[ n_+ = 3, \quad n_- = 4 \quad \Rightarrow \quad \text{ind } D[A] = n_- - n_+ = 1 \]

Compare \( Q[A] = 0 \)
After non-periodic gauge transformations at $t=1$:

- **Index of overlap operator:**
  
  
  \[
  n_+ = 1, \quad n_- = 0 \quad \implies \quad \text{ind } D[A] = n_- - n_+ = -1
  \]

  Compare $Q[A] = 0$
Two Thick Spherical SU(2)-vortices, $40^3 \times 2$-lattice

One vortex at $t = 1$ and another at $t = 2$

$y = 7$, $t = 1$, $\chi = 0$, $n = 1 - 4$, $\max = 0.0000314321$ $y = 7$, $t = 1$, $\chi = 0$, $n = 5 - 5$, $\max = 0.00315628$

Polyakov loop

Index of overlap operator:

$n_+ = 3$, $n_- = 5$

$\text{ind } D[A] = n_- - n_+ = 2$

Compare $Q[A] = 0$
One vortex at $t = 1$ and another at $t = 2$

Index of overlap operator:

\[ n_+ = 0, \quad n_- = 0 \quad \Rightarrow \quad \text{ind} \ D[A] = n_- - n_+ = 0 \]

Compare $Q[A] = 0$

**Conclusion:** lattice index theorem inapplicable?
Conclusions

- Index theorem fulfilled for U(1) vortices
- Index theorem puzzeling for SU(2) vortices

This is likely due to the discontinuous nature of $U_t$ in the continuum limit, near the vortex center, as function of $t$.

For all those configurations, Overlap Fermions give the same topological charge as cooling.
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Thank you for your attention!

Questions?