Irreducible three-quark operators for LQCD

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Motivation: Distribution Amplitudes

future aim

- Distribution Amplitude of a Proton $\phi_P$
- need: suitable operators $O$ for $\langle 0 | O | P \rangle$
- especially: controlling mixing!

mixing under renormalization

- incorporating radiative corrections modifies operators
- easiest case: $O^R = ZO$
- more general, there might occur mixing:

$$O_i^R = \sum_j Z_{ij} O_j$$

- this talk: nucleon operator mixing and renormalization
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- this talk: nucleon operator mixing and renormalization
controlling mixing

- a typical 3q operator:
  \[ O_i = T_{\alpha\beta\gamma\mu}^{(i)} \cdot u_\alpha u_\beta D_\mu d_\gamma \]

- problematic: mixing
  \[ O_i^R = \sum_j Z_{ij} O_j \]

- excluding mixing by space-time symmetries of \( O_j \)
- fewer symmetries on lattice \( \Rightarrow \) more mixing

hence

- study transformation of 3q operators under space group of the hypercubic lattice \( H_4 \)
Renormalisation and mixing

controlling mixing

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The symmetry group $\tilde{H}_4$

Spinorial group $\tilde{H}_4$ \(^1\)

- Group theoretical approach
- Symmetry group for the hypercubic lattice
- Describing spinors: set of equations for six generators

\[
\begin{align*}
I_i^2 &= -1, & I_i I_j &= -I_j I_i \\
\gamma^2 &= -1, & t^3 &= -1, & (t\gamma)^4 &= -1 \\
\gamma I_1 &= -I_3 \gamma, & \gamma I_2 &= -I_2 \gamma, & \gamma I_4 &= -I_4 \gamma \\
t I_1 &= I_1 t, & t I_2 &= I_4 t, & t I_3 &= I_2 t, & t I_4 &= I_3 t
\end{align*}
\]

- $-1$: phase for 360°-rotation

\(^1\)Dai, Song (2001)
Representations for \( \bar{H}_4 \)

represents and mixing

- representation: defines how an object transforms under group action
representations and mixing

- representation: defines, how an object transforms under group action

\[ \tau^4_1 \text{ matrices: transformation for a derivative } D_\mu \]
\[ \tau^4_1 \text{ matrices: transformation for a quark-field } u_\alpha \]

\[
\gamma = \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & -1 & -1 \\
1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0
\end{pmatrix}
\]
\[
t = \frac{1}{2} \begin{pmatrix}
1 - i & -1 - i & 0 & 0 \\
1 - i & 1 + i & 0 & 0 \\
0 & 0 & 1 + i & 1 - i \\
0 & 0 & -1 - i & 1 - i
\end{pmatrix}
\]

\[
l_1 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & i & 0 \\
i & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}
\]
\[
l_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]
\[
l_3 = \begin{pmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]
\[
l_4 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
representations and mixing

- representation: defines, how an object transforms under group action
- $\Rightarrow$ transformation of nucleon operators
- no mixing between inequivalent irreducible representations

strategy

- construct irreducibly transforming three-quark operators
- from rotations $SO_4$ via continuum $O_4$ to lattice $\bar{H}_4$
Irreducible representations in $SO_4$

quark-fields

- Weyl spinor: $SU(2) \times SU(2) \simeq SO(4)$

$$u_\alpha = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \{ u^a, (+) \}$$

$$u_\alpha u_\beta d_\gamma: \quad +++, ++-, +--$$

covariant derivatives

- contract with Pauli-matrices $D_\mu \rightarrow (D_\mu \sigma^\mu)^{d_\dot{e}}$

$SO_4$ irreducible representation:

- independently symmetrize $(+)$ and $(-)$ indices
from $SO_4$ to $O_4$

- reflections $r$: $O_4 = SO_4 \cup rSO_4$
- i.e. $O_4$ is closed under parity $P$

from $O_4$ to lattice $\bar{H}_4$

- given a set of $O_4$ irreducible operators and all their $\bar{H}_4$ transformation matrices
- construct a projector $P^\alpha$ to decompose w.r.t. $\bar{H}_4$

final result

- subspace of $\bar{H}_4$-irreducible operators of $\tau^\alpha$
Typical mixing-controlled 3q operators

irreducible representation $\tau^{12}_1$

\begin{align*}
\mathcal{O}^{D-+++}_{121,1} &= 3u_2u_3(D_3 - iD_4)d_3 + u_1u_4(D_3 + iD_4)d_3 \\
&+ u_1u_3(D_3 + iD_4)d_4 - u_1u_4(D_1 + iD_2)d_4 \\
\mathcal{O}^{D-+++}_{121,2} &= +u_2u_3(D_1 - iD_2)d_3 + u_2u_4(D_3 - iD_4)d_3 \\
&+ u_2u_3(D_3 - iD_4)d_4 + 3u_1u_4(D_3 + iD_4)d_4 \\
\vdots \\
\mathcal{O}^{D-+++}_{121,12} &= -u_3u_1(D_1 - iD_2)d_1 - u_4u_1(D_3 - iD_4)d_1 \\
&- u_3u_1(D_3 + iD_4)d_3 - u_3u_2(D_3 + iD_4)d_1 \\
&+ u_4u_1(D_1 + iD_2)d_2 + u_4u_2(D_1 + iD_2)d_1
\end{align*}
\[ O_{122a,1}^{DD++} = + u_1 u_1 D_1 D_1 d_1 + u_2 u_2 D_1 D_1 d_1 + 2i u_1 u_1 D_2 D_1 d_1 \\
- u_2 u_1 D_3 D_1 d_1 - u_1 u_2 D_3 D_1 d_1 + 2i u_1 u_1 D_2 D_2 d_1 \\
- u_1 u_1 D_2 D_2 d_1 - u_2 u_2 D_2 D_2 d_1 - u_2 u_1 D_4 D_2 d_1 \\
- u_1 u_2 D_4 D_2 d_1 - u_2 u_1 D_1 D_3 d_1 - u_1 u_2 D_1 D_3 d_1 \\
- u_1 u_1 D_3 D_3 d_1 - u_2 u_2 D_3 D_3 d_1 - 2i u_1 u_1 D_4 D_3 d_1 \\
- u_2 u_1 D_2 D_4 d_1 - u_1 u_2 D_2 D_4 d_1 - 2i u_1 u_1 D_3 D_4 d_1 \\
+ u_1 u_1 D_4 D_4 d_1 + u_2 u_2 D_4 D_4 d_1 + u_2 u_1 D_1 D_1 d_2 \\
+ u_1 u_2 D_1 D_1 d_2 - u_1 u_1 D_3 D_1 d_2 + u_2 u_2 D_3 D_1 d_2 \\
- u_2 u_1 D_2 D_2 d_2 - u_1 u_2 D_2 D_2 d_2 - u_1 u_1 D_4 D_2 d_2 \\
+ u_2 u_2 D_4 D_2 d_2 - u_1 u_1 D_1 D_3 d_2 + u_2 u_2 D_1 D_3 d_2 \\
- u_2 u_1 D_3 D_3 d_2 - u_1 u_2 D_3 D_3 d_2 - u_1 u_1 D_2 D_4 d_2 \\
+ u_2 u_2 D_2 D_4 d_2 + u_2 u_1 D_4 D_4 d_2 + u_1 u_2 D_4 D_4 d_2 \]
**Sorting irreducible operators**

**controlled mixing**
- only operators of same representation and same or lower dimension can mix

<table>
<thead>
<tr>
<th></th>
<th>(mass)dimension 9/2</th>
<th>dimension 11/2</th>
<th>dimension 13/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{1}^{4}$</td>
<td>$\mathcal{O}<em>{41}^{[++++]}$, $\mathcal{O}</em>{41}^{(++++)}$, $\mathcal{O}<em>{41}^{[++--]}$, $\mathcal{O}</em>{41}^{[---]}$, $\mathcal{O}_{41}^{[+-+-]}$</td>
<td>$\mathcal{O}<em>{41}^{DD++-}$, $\mathcal{O}</em>{41}^{DD+-+}$, $\mathcal{O}_{41}^{DD---}$</td>
<td></td>
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<tr>
<td>$\tau_{2}^{4}$</td>
<td>$\mathcal{O}<em>{8}^{[+++]}$, $\mathcal{O}</em>{8}^{D+++}$</td>
<td>$\mathcal{O}<em>{8}^{DD++-}$, $\mathcal{O}</em>{8}^{DD+-+}$, $\mathcal{O}_{8}^{DD---}$</td>
<td></td>
</tr>
<tr>
<td>$\tau_{1}^{8}$</td>
<td>$\mathcal{O}<em>{121}^{[+-+]}$, $\mathcal{O}</em>{121}^{[+++]}$, $\mathcal{O}<em>{121}^{D++-}$, $\mathcal{O}</em>{121}^{D-+-}$</td>
<td>$\mathcal{O}<em>{121}^{DD++-}$, $\mathcal{O}</em>{121}^{DD+-+}$, $\mathcal{O}_{121}^{DD---}$</td>
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<tr>
<td>$\tau_{2}^{12}$</td>
<td>$\mathcal{O}<em>{122a}^{D+++}$, $\mathcal{O}</em>{122b}^{D+++}$, $\mathcal{O}<em>{122}^{DD++-}$, $\mathcal{O}</em>{122}^{DD+-+}$, $\mathcal{O}_{122}^{DD---}$</td>
<td></td>
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</tbody>
</table>
Renormalization on the lattice

renormalization: what to compute

- isospin symmetrized, irreducible 3q operator
e.g.

\[ \mathcal{O}^{(i)}(z) = \epsilon_{c_4c_5c_6} T^{(i)}_{\alpha\beta\gamma} \cdot (u_{\alpha c_4} d_{\beta c_5} u_{\gamma c_6}(z) - d_{\alpha c_4} u_{\beta c_5} u_{\gamma c_6}(z)) \]

- contracted with three quark sources

- matrix element \( G^\mathcal{O} \) in Landau gauge:

\[ G^\mathcal{O}(p, q, r)_{\alpha\beta\gamma c_1 c_2 c_3} = FT \langle \bar{u}(u)_{\alpha c_1} \bar{u}(v)_{\beta c_2} d(w)_{\gamma c_3} \cdot \mathcal{O}^{(i)}(z) \rangle \]
extracting the 3q vertex $\Gamma$

- amputate external legs

$$G^O(p, q, r)_{\alpha\beta\gamma c_1 c_2 c_3} = \Gamma^O(p, q, r)_{\alpha'\beta'\gamma'} S(-r)_{\alpha'\alpha} S(-p)_{\beta'\beta} S(-q)_{\gamma'\gamma} \epsilon_{c_1 c_2 c_3}$$

renormalization

- from regulated to renormalized quantities:

$$\Gamma^\text{ren}_i = Z^O_{ij} \cdot Z_q^{-3/2} \Gamma_j$$

- 3q operator renormalization matrix $Z^O_{ij}$

- matching with (1 loop) continuum PT: $Z^{\overline{MS}}(2\text{GeV})$
Preliminary results

scaling and extrapolation with RGE: typical $Z_{ii}^{\overline{MS}}$

\[ Z(\mu) \]

\[ Z(2\text{GeV}) \]

\( \mu^2/\text{GeV}^2 \)

Th. Kaltenbrunner (QCDSF)

irreducible 3q operators for LQCD

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moments of nucleon distribution amplitudes

- Niko’s talk: $\Phi^{100} + \Phi^{010} + \Phi^{001} = \Phi^{000} = 3$
- relates operators without and with one derivative
- blue: bare, red: renormalized sum
Summary and outlook

discussed here
- mixing of 3q operators
- spinorial hypercubic group
- irreducible representations

in progress
- lattice renormalization
- detailed analysis of results
- use with DAs for the proton