Pion form factor from all-to-all propagators of overlap quarks

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1. introduction

**pion form factor**

- confinement $\Rightarrow$ reactions of hadrons $\ni$ form factors $\Leftarrow$ lattice QCD

- normalization $F(q_{\text{ref}}^2)$, charge radius $\langle r^2 \rangle$
  $\Leftarrow$ need accurate data (at finite meson momentum)
  - twisted boundary condition (Bedaque, 2004)
  - low-mode averaging (Giusti et al., 2004; DeGrand-Schaefer, 2004)
  - all-to-all quark propagator (TrinLat, 2005)
  - ...

- pion electromagnetic form factor $F_\pi(q^2)$:
  - pion: plays central role in low-energy QCD
  - simple structure : CVC $\Rightarrow F_\pi(0) = 1$, ...
  $\Rightarrow$ useful for testing new methods
1. introduction

this (on-going) work

calculate pion form factor in $N_f = 2$ QCD from all-to-all quark propagator

- all-to-all propagator: precise determination of $F_\pi(q^2)$
- overlap quarks + small $m_{\text{sea}}$: chiral behavior of $\langle r^2 \rangle$

outline

- simulation parameters
  - lattice action, $a$, $m_{\text{sea}}$, ...

- pion correlators from all-to-all propagators
  - all-to-all propagators, pion correlators

- pion form factor $F_\pi$, charge radius $\langle r^2 \rangle$
  - $q^2$ dependence of $F_\pi$, chiral extrap. of $\langle r^2 \rangle$

- summary
2. simulation method

- $N_f = 2$ QCD w/ degenerate $u,d$ quarks
- overlap quark action w/ std. Wilson kernel

\[ S_q = \sum \bar{q} D_{ov} q, \quad D_{ov} = \left( m_0 + \frac{m}{2} \right) + \left( m_0 - \frac{m}{2} \right) \gamma_5 \text{sgn}[H_w(-m_0)], \quad m_0 = 1.6 \]

- Iwasaki gauge action $\Leftarrow$ study of low-mode density of $H_w$, locality
- determinant to suppress zero modes $\det[H_w^2]/\det[H_w^2 + \mu^2]$ $(\mu = 0.2)$
  - does NOT change continuum limit
  - need to study effects of fixed topology (Brower et al., 2003; Aoki et al., 2007)

- $\beta = 2.30$: $a = 0.1184(16)$ fm $\Leftarrow$ $r_0 = 0.49$ fm
- $16^3 \times 32$: $L \sim 1.9$ fm

- data in this talk:
  - $4 m_{\text{sea}}$: $m_{\text{sea}} = m_{s,\text{phys}}/6 - m_{s,\text{phys}}/2$, $M_{\text{PS}} = 288 - 521$ MeV
  - 50 conf $\times$ 100 HMC traj. = 5,000 traj.

- in $Q = 0$ sector

overview / details on simulation method $\Rightarrow$ plenary talk by H. Matsufuru
3.1 all-to-all quark propagators

construction of all-to-all propagators

follow strategy proposed by TrinLat collaboration \cite{Foley:2005}(Foley et al., 2005)

low-mode contribution

- exact: from eigenpairs of $D_{ov}$
- implicitly restarted Lanczos $\Rightarrow 100(= N_{ep})$ low-lying modes of $D_{ov}$

\[
(D_{ov}^{-1})_{low} = \sum_{k=1}^{N_{ep}} \frac{1}{\lambda(k)} u(k)^{\dagger}
\]

high mode contribution

- stochastic: noise method + dilution
- noise method: one $Z_2$ noise / conf ($N_r = 1$)
- dilution: support on a subset of time/spin/color
  in this study: color $\times 1$, spinor $\times 1$, time $\times 2$ $\Rightarrow N_d = 3 \times 4 \times (N_t/2)$
- solve linear eq. for $N_r \times N_d$ noise vectors

\[
D_{ov} x^{(r,d)} = (1 - P_{low}) \eta^{(r,d)} \quad \Rightarrow \quad (D_{ov}^{-1})_{high} = \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{d=1}^{N_d} x^{(r,d)} \eta^{(r,d)}
\]
3.1 all-to-all quark propagator

in summary, all-to-all quark propagator $D_{ov}^{-1}$ can be expressed as follows:

$$D_{ov}^{-1} = \sum_{k=1}^{N_{vec}} (v^{(k)} w^{(k)\dagger})_{ij} \quad (N_{vec} = N_{ep} + N_r N_d)$$

$$v^{(k)} = \left\{ u^{(1)}, \ldots, u^{(N_{ep})}, x^{(1,1)}, \ldots, x^{(N_r, N_d)} \right\}$$

$$w^{(k)} = \left\{ \frac{u^{(1)}}{\lambda^{(1)}} \, \ldots, \frac{u^{(N_{ep})}}{\lambda^{(N_{ep})}}, \frac{\eta^{(1,1)}}{N_r} \, \ldots, \frac{\eta^{(N_r, N_d)}}{N_r} \right\}$$
2-pt. functions

\[ C_{\Gamma\Gamma'}(t, t'; p) = \langle (\bar{g} \Gamma f)(t') (\bar{f} \Gamma g)(t) \rangle \text{ (with } p) = \sum_{k=1}^{N_{\text{vec}}} \sum_{l=1}^{N_{\text{vec}}} \mathcal{O}_{\Gamma, k}^{(l, k)}(t'; p) \mathcal{O}_{\Gamma, l}^{(k, l)}(t; -p) \]

\[ \mathcal{O}_{\Gamma, l}^{(k, l)}(t; p) = \sum_{r} \phi^{(r)}(r) w(x + r, t)(k)^{\dagger} \Gamma v^{(l)}(x, t) \exp[-ipx] \]

\[ v^{(k)}, w^{(k)} : \text{ can be re-used for different } \phi^{(r)} \text{ and } p \]
3.2 meson correlators

test on pion 2-pt. functions
comparison with conventional method

- at $m_{\text{sea}} \approx m_{s,\text{phys}}/4$; local source/sink
- conventional method: average over sink location, fixed source
- consistent with conventional method
- smaller statistical fluctuation at large $t$
3.2 meson correlators

3-pt. functions

\[ C_{\Gamma\Gamma'\Gamma''}(t, t', t''; \mathbf{p}, \mathbf{p}') = \sum_{k=1}^{N_{\text{vec}}} \sum_{l=1}^{N_{\text{vec}}} \sum_{m=1}^{N_{\text{vec}}} \mathcal{O}^{(m,l)}_{\Gamma'}(t'; \mathbf{p}') \mathcal{O}^{(l,k)}_{\Gamma''}(t''; \mathbf{p} - \mathbf{p}') \mathcal{O}^{(k,m)}_{\Gamma}(t; -\mathbf{p}) \]

- \( v^{(k)}, w^{(k)} \): can be re-used for different \( \phi^{(i)}, \mathbf{p}^{(i)} \)
- \( \mathcal{O}^{(k,l)}_{\Gamma}(t; \mathbf{p}) \): can re-use those for 2-pt. functions
- can take any combination of \( t, t', t'' \)
3.3 measurement details

on IBM BG/L @ KEK (10 racks, 57.3 TFLOPS)

- Calculation of eigenpairs
  - 100 pairs from implicitly restarted Lanczos
  - 3 TFLOPS \cdot hours /100 pairs /conf

- Quark propagators with diluted noise sources
  - Conventional: 12 (=color \times spinor) inversions /source (=smearing) /conf
  - This study: 12 \times N_t/2 inversions /source /conf
  - \Rightarrow more expensive by a factor of N_t/2 = 16 ?
  - \Leftarrow a factor of 8 speed up by low-mode preconditioning w/ 100 eigenpairs
  - 9 TFLOPS \cdot hours / conf

on SR11K @ KEK (16 nodes, 2.15 TFLOPS)

- Construction of meson operator $\mathcal{O}_\Gamma$
  - 13 GFLOPS \cdot hours /m_{val} /p /16 $\Gamma$’s /conf

and construction of 2- and 3-pt. functions on PCs, SR11K
3.3 measurement details

- can average over meson operator locations
  - spatial location of source $\mathbf{x}$
  - temporal location $t$ (with $\Delta t$ and $\Delta t'$ fixed)

- does NOT need extra inversion $D_{ov}^{-1}$ for different smearing function / meson momentum
  - employ local and exponentially smeared operators
    $$\phi(|\mathbf{r}|) = \exp[-0.4 |\mathbf{r}|] \iff \text{effective mass plot for } \pi$$
  - take 33 meson momenta w/ $|\mathbf{p}| \leq 2$ ($p_k$ in unit of $2\pi/L$ in this talk)
  - 11 values for $q^2$

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3.3 measurement details

fluctuation of jackknife data

w/o averaging

[Graph showing jackknife data without averaging]

averaged

[Graph showing jackknife data with averaging]

- “jackknife data”/average for $C_{PV4P}$ at $\Delta t = \Delta t' = N_t / 4$, $|p| = \sqrt{2}$, $|p'| = 0$
- compare
  - data w/o averaging
  - data averaged over source op. location / momentum configurations
- “all-to-all” is effective also for 3-pt. functions
4.1 a ratio method

\[ C_{4,\phi}(\Delta t, \Delta t'; p, p') \rightarrow \frac{\sqrt{Z_{\pi,\phi}(|p|) Z_{\pi,\phi}(|p'|)}}{4E(p)E(p')Z_V} e^{-E(p)\Delta t} e^{-E(p')\Delta t'} \langle \pi(p') | V_4 | \pi(p) \rangle \]

\[ C_{\phi,\phi'}(\Delta t; p) \rightarrow \frac{\sqrt{Z_{\pi,\phi}(|p|) Z_{\pi,\phi'}(|p'|)}}{2E(p)} e^{-E(p)\Delta t}, \quad \sqrt{Z_{\pi,\phi}(|p|)} = \langle \pi(p) | O_{\pi,\phi}(p)^{\dagger} \rangle \]

\[ R(\Delta t, \Delta t'; p, p') = \frac{C_{\pi,\phi,\phi}(\Delta t, \Delta t'; p, p')}{C_{\phi,\text{lcl}}(\Delta t; p) C_{\text{lcl},\phi}(\Delta t'; p')} = \frac{1}{\sqrt{Z_{\pi,\text{lcl}} Z_{\pi,\text{lcl}} Z_V}} \langle \pi(p') | V_4 | \pi(p) \rangle \]

\[ F_{\pi}(q^2) = \frac{2M_{\pi}}{E(p) + E(p')} \frac{R_{\mu,\phi,\phi}(\Delta t, \Delta t'; p, p')}{R_{4,\phi,\phi}(\Delta t, \Delta t'; 0, 0)} \]
4.1 a ratio method

**effective plot of** $F_\pi(q^2)$

- **all-to-all** ⇒ can take any combination of $(\Delta t, \Delta t')$
- choose combinations $(\Delta t, \Delta t')$ from
  - plot as a function of $\Delta t$ with sufficiently large $\Delta t, \Delta t'$
  - plot with fixed $\Delta t + \Delta t'$ (conventional analysis)
- $F_\pi(q^2)$ from constant fit

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4.2 $q^2$ dependence

- Small difference from VMD $\Leftarrow \rho$ bound-state below $\pi \pi$ threshold (?)

$$F_\pi(q^2) = \begin{cases} 1 - \frac{q^2}{M_\rho^2} + c_1 q^2 + c_2 (q^2)^2 + \cdots \\ \frac{c}{1-q^2/M_\rho^2} + \frac{c'}{1-c'' q^2} + \cdots \end{cases}$$

- Reasonable $\chi^2$/dof $\sim 1$

- Fit curves consistent w/ each other
4.3 charge radius

\[ F_\pi(q^2) = 1 - \frac{1}{6} \langle r^2 \rangle q^2 + \ldots \]

- stable against \( q^2 \)-fit form
- stable against \( (q^2)_{\text{min}} \)
- central value
  \( \Leftarrow \) use \( M_\rho \) pole + quad
- sys.err.
  \( \Leftarrow \) variation of \( (q^2)_{\text{min}} \), fit form

- leading finite volume correction: \( \sim 3\% \) at lightest \( m_{\text{sea}} \); included
  (Bunton et al., 2006; Chen et al., 2006; Borasoy-Lewis et al., 2005; QCDSF-UKQCD, 2006)
4.3 charge radius

Charge radius

\[ \langle r^2 \rangle = c_0 - \frac{1}{(4\pi f_0)^2} \ln \left[ \frac{M_\pi^2}{\mu^2} \right] + c_1 M_\pi^2 \]

- \( f_0 \) from spectrum \( \text{\textit{(talk by Noaki)}} \)
- from \( \epsilon \)-regime \( \text{\textit{(talk by Fukaya)}} \)
- \( \mu = 4\pi f_0 \)
- no large deviation from ChPT form \( (\chi^2/\text{dof} \sim 1.2) \)

\[ \langle r^2 \rangle = 0.388(9)_{\text{stat}}(12)_{\text{sys}} \text{fm}^2 \]

\( \langle r^2 \rangle \) held to be

\[ \left\{ \begin{array}{l}
\text{scale ambiguity} : \Delta(\langle r^2 \rangle) \propto 2\Delta a \\
\text{finite size effects} (\exists \text{ fixed topology}) \\
\text{larger } L/a; \text{ smaller } m_{\text{sea}} : M_\rho > \pi\pi \text{ threshold}
\end{array} \right. \]
5. summary

measurement using all-to-all quark propagators
⇒ pion form factor in $N_f = 2$ QCD with dynamical overlap quarks

- all-to-all propagator
  - does NOT need additional inversion $D_{ov}^{-1}$ for different meson momentum
  ⇒ useful for \{overlap simulations, (PS meson) form factors\}

- pion form factor / charge radius
  - all-to-all ⇒ clear signal in wide region of $q^2$
  - charge radius $\langle r^2 \rangle = 0.388(9)(12)$ fm$^2$: need further studies of systematics
    - effect of fixed topology, uncertainties in lattice scale, ...

- future directions
  - scalar form factor ⇒ scalar radius (on-going)
  - $K \rightarrow \pi$ decays, ... (on-going)
  - flavor singlet physics (soon)
  - on larger volume(s) in $N_f = 3$ (planning)
6.1 a ratio method

**Meson energy** $E(p)$

- **Average over momentum** $\Rightarrow$ clear signal for $|p| \leq \sqrt{3}$
- Good agreement w/ (continuum) dispersion relation
- This study: use fitted energy from smr-smr correlator $\Rightarrow$ to avoid possible underestimation of error
6.2 meson correlators

on pion / $A_4$ / vector correlators

comparison of LL/LH/HL/HH mode contributions

$$\langle O_\Gamma O_\Gamma^\dagger \rangle = D^{-1}_{\text{low}} \otimes D^{-1}_{\text{low}} + D^{-1}_{\text{low}} \otimes D^{-1}_{\text{high}} + D^{-1}_{\text{high}} \otimes D^{-1}_{\text{low}} + D^{-1}_{\text{high}} \otimes D^{-1}_{\text{high}}$$

- $\langle P P^\dagger \rangle$: dominated by LL contribution at $t \geq 4$
  - can use small $N_r$
- $\langle A_4 A_4^\dagger \rangle$, $\langle V V^\dagger \rangle$: significant LH/HL/HH contributions at $t \leq 8$
- this study: use $P$ for pion interpolation field ⇔ some of previous studies: $A_4$