Meson form factors using four point functions

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Introduction

Study of hadron structure using four point functions

- Inserting the density operator at equal times
  → Hadron wave function

- Inserting the electromagnetic current operator at different times
  → Hadron form factors
Four point functions

Requires a summation over spacial coordinates of the sink and current

\[ G_{h}^{j_{\sigma}}(\vec{y}, t_1, t_2, t) = \int d^3xd^3z \langle h(\vec{z}, t) | j_{\sigma}^{\mu}(\vec{x} + \vec{y}, t_2) j_{\sigma}^{d}(\vec{x}, t_1) | h(\vec{0}, 0) \rangle \]

we exclude disconnected diagrams

\[ \sum_{\vec{x}} \text{sets } \vec{p}_{\text{sink}} = \vec{p}_{\text{source}} \quad \mid \quad \sum_{\vec{z}} \text{sets } \vec{p}_{\text{sink}} = \vec{p}_{\text{source}} = \vec{0} \]

to accomplish both summations one requires the all - to - all propagator

\[ \rightarrow \text{use of stochastic techniques to obtain the all - to - all propagator} \]
Liverpool “One end trick”*

Decrease stochastic noise by combining two solution vectors appropriately.

- E.g.

\[
\sum_{\vec{x}} \left< \phi(x)_{\mu}^a \phi^*(x)_{\mu}^a \right>_r = \sum_{\vec{x}, \vec{x}_0, \vec{y}_0} M^{-1}(x; x_0)_{\mu \nu}^a b^b c^c \delta(\vec{x}_0 - \vec{y}_0) \delta_{bb'} \delta_{vv'} = \\
\sum_{\vec{x}, \vec{x}_0} \text{Tr} \left( |M^{-1}(x; x_0)|^2 \right)
\]

is the \( \pi \) two point function summed over all source coordinates.

- For arbitrary interpolating operators use spin dilution:

\[
\eta^a_{\mu r}(x) = \eta^a(x) \delta_{\mu r}
\]

**Downside:** Need a new set of solution vectors for each momentum

A first time application of the “One end trick” to four point functions

Since the four point function is needed at $\vec{p}_{\text{sink}} = \vec{p}_{\text{source}} = 0$ there's no downside in using the “One end trick”:

$$G^j_\Gamma(\vec{y}; t_1, t_2, t) = \sum_{\vec{x}} Tr \left[ S^{[\sigma]}(\Gamma; \vec{x}, t_1; t; t_0) S^{[\sigma]}(\bar{\Gamma}; \vec{x} + \vec{y}, t_2; t_0; t) \right]$$

where: $S^{[\sigma]}(\Gamma; x; t; t_0) = (\Gamma \gamma_5)_{\mu\kappa} \phi^a_v(x; t) \gamma_5 \gamma_\sigma \phi^a_{\mu'}(x; t_0)$

The Fourier transform of $G^j_\Gamma(\vec{y}; t_1, t_2, t)$ gives the form factor squared for all momenta

Two sets of solution vectors: \[
\begin{cases}
\phi(x; t_0) & \text{from source} \\
\phi(x; t) & \text{from sink}
\end{cases}
\]
Hadron wave functions

Taking $t_1 = t_2$, in the non-relativistic limit, the density-density correlator reduces to the wave function squared

$$j^0(\vec{x} + \vec{y}, t_1)$$

\[
\begin{array}{c}
(\vec{x}, t) \\
\times
\end{array}
\rightarrow
\begin{array}{c}
(\vec{0}, 0) \\
\times
\end{array}
\rightarrow |\Psi_h(\vec{y})|^2
\]

\[
\begin{array}{c}
(\vec{x}, t_1)
\end{array}
\]

Parameters:

<table>
<thead>
<tr>
<th># Conf.</th>
<th>$\kappa$</th>
<th>$a m_\pi$</th>
<th>$m_\pi/m_\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24$^3 \times 40^*$</td>
<td>185</td>
<td>0.1575</td>
<td>0.270(3)</td>
</tr>
<tr>
<td>150</td>
<td>0.1580</td>
<td>0.199(3)</td>
<td>0.56</td>
</tr>
<tr>
<td>24$^3 \times 32^*$</td>
<td>200</td>
<td>0.15825</td>
<td>0.150(3)</td>
</tr>
</tbody>
</table>

- Source - sink separation: $t/a = 14$
- Density insertions at: $t_1/a = 7$
- Wuppertal and HYP smearing on initial/final states

*SESAM collaboration (T$\chi$L), B. Orth et al., Phys. Rev. D72(2005)014503
Hadron wave functions

Improvement using “One end trick”:

\( \rho \) meson wave function

- **Naive calculation:**
  24 color - spin, even - odd diluted noise vecs. \( \Rightarrow \) 144 inversions

- **One end trick:**
  8 spin diluted noise vecs.
  \( \Rightarrow \) 32 inversions
Hadron wave functions

$\rho_0$ meson asymmetry:

Clear asymmetry in $z$ projection of the $\rho$ meson's wave function
Meson form factors

The Fourier transform of the four point function, at large time separation \( t_2 - t_1 \), is associated with the meson's form factor:

\[
\sum_{s'} |\langle H_F | h(s; 0) \rangle|^2 \frac{|\langle h(s; 0) | j_\sigma | h(s'; p) \rangle|^2}{8M^2E(p)} e^{-E(p)(t_2-t_1)} e^{-M(t-(t_2-t_1))}
\]

in particular for the pion:

\[
G^{j^\sigma}_{\gamma_5}(\vec{p}; t_1, t_2, t) = \left| \langle H_{\gamma_5} | \pi(0) \rangle \right|^2 \frac{|\langle \pi(0) | j_\sigma | \pi(\vec{p}) \rangle|^2}{8M^2E(p)} e^{-E(p)(t_2-t_1)} e^{-M(t-(t_2-t_1))}
\]

\[
= \left| \langle H_{\gamma_5} | \pi(0) \rangle \right|^2 \frac{|(p^\sigma + p_0^\sigma)F_\pi(Q^2)|^2}{8M^2E(p)} e^{-E(p)(t_2-t_1)} e^{-M(t-(t_2-t_1))}
\]

\[
p = (E, \vec{p}), \quad p_0 = (M, \vec{0}), \quad Q^2 = (p - p_0)^2
\]
π form factor

Divide four point functions with two point functions and search for plateau vs. $t_2 - t_1$:

$$R_{Y_5}^{j_0}(\vec{p}; t_1, t_2) = \frac{\sqrt{4E(p)M}}{E(p) + M} \left( \frac{G_{Y_5}^{j_0}(\vec{p}; t_1, t_2, t)G_{Y_5}(\vec{p}, t_1)}{G_{Y_5}(\vec{p}, t_2)G_{Y_5}(\vec{0}, t - (t_2 - t_1))} \right)$$

$G_{Y_5}(\vec{0}, t)$: One end two point function (zero momentum only)

$G_{Y_5}(\vec{p}, t)$: Standard two point function (finite momentum)

- $t_1/a$ fixed at 3
- two sets: $t_{\text{sink}}/a = 14$ and 16 for consistency check
- set:
  - $t = 16/a$ for $24^3 \times 40$ lattice
  - $t = 14/a$ for $24^3 \times 32$ lattice
π form factor

$F_\pi(Q^2)$ at three $\kappa$ values compared to vector meson dominance (VMD):

VMD: $\frac{1}{1+Q^2/m_\rho^2}$ at $m_\rho = 0.77\text{GeV}$ (physical)
π form factor

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† Parallel talk by S. Simula
\textbf{\( \rho \) form factors}

\( \rho \) meson form factors can also be extracted by appropriate combinations of \( \rho \) polarizations and current directions:

\[
G_{\rho k}^j(\bar{p}_\perp; t_1, t_2, t) = \frac{\hat{f}^2(\vec{0})}{8M^2E(p_\perp)} (E(p_\perp) + M)^2 G_1^2(Q^2)e^{-M(t-(t_2-t_1))}e^{-E(p_\perp)(t_2-t_1)}
\]

\[
G_{\rho k}^j(\bar{p}_\perp; t_1, t_2, t) = \frac{\hat{f}^2(\vec{0})}{8M^2E(p_\perp)} \left(E^2(p_\perp) - M^2\right) G_2^2(Q^2)e^{-M(t-(t_2-t_1))}e^{-E(p_\perp)(t_2-t_1)}
\]

where:

\[
\langle \Omega | \chi_{\gamma k} | \rho(\bar{p}, s) \rangle = \hat{\rho}(\vec{p}) \epsilon_k(\bar{p}, s), \quad \sum_s \epsilon_k(\bar{p}, s) \epsilon^*_k(\bar{p}, s) = g_{kk'} - \frac{p_k p_{k'}}{M^2}
\]

and \( \bar{p}_\perp \) is a momentum perpendicular to the \( k \) direction.

\( G_1, G_2 \) and \( G_3 \) are associated to the physical from factors through:

\[
G_Q = G_1 - G_2 + \left(1 + \frac{Q^2}{4M^2}\right) G_3 \quad \text{extraction of } G_3 \text{ not as straight forward}
\]

\[
G_M = G_2
\]

\[
G_C = G_1 + \frac{2}{3} \frac{Q^2}{4M^2} G_Q
\]
Preliminary results for $\rho$ form factors: $G_1(Q^2)$ and $G_2(Q^2)$

$m_\rho = 1.002(41)$ GeV
$m_\rho = 0.910(38)$ GeV
$m_\rho = 0.853(37)$ GeV

Analysis for $G_3$ in progress to extract physical quantities (e.g. $\mu_\rho$, $Q_\rho$, etc.)
First application of stochastic “One end trick” for the computation of meson four point functions

- Very accurate results for hadron wave functions
  - $\rho$ is clearly a prolate

- Application to form factor calculation
  - $\pi$ form factor extracted, good results compared with other methods
  - $\rho$ form factor calculation in progress, reasonable preliminary results