$K \to \pi\pi$ Amplitudes at Unphysical Kinematics Using Domain Wall Fermions

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Introduction

- $K \rightarrow \pi\pi$ decays on the lattice are interesting because the typical energies involved are less than $\Lambda_{QCD}$ so that QCD effects are important in this decay.

- The direct CP violating parameter $\epsilon'/\epsilon$ can be found from $K \rightarrow \pi\pi$ calculations.

- The precision of domain wall fermions (DWF) on a $24^3 \times 64$, $L_s = 16$ lattice are needed to get reasonable uncertainties.
Effective Hamiltonian

The weak interactions are included in an effective Hamiltonian

\[ \mathcal{H}_{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_i V_{CKM}^i c_i(\mu) Q_i \]

where \( c_i(\mu) \) are the Wilson coefficients and \( Q_i \) are four quark operators, for example

\[ Q_1 = \bar{s}_a \gamma_\mu (1 - \gamma^5) d_a \bar{u}_b \gamma^\mu (1 - \gamma^5) u_b \]
\[ Q_2 = \bar{s}_a \gamma_\mu (1 - \gamma^5) d_b \bar{u}_b \gamma^\mu (1 - \gamma^5) u_a \]
\[ Q_9 = \frac{3}{2} \bar{s}_a \gamma_\mu (1 - \gamma^5) d_a \sum_q e_q \bar{q}_b \gamma^\mu (1 - \gamma^5) q_b \]
\[ Q_{10} = \frac{3}{2} \bar{s}_a \gamma_\mu (1 - \gamma^5) d_b \sum_q e_q \bar{q}_b \gamma^\mu (1 - \gamma^5) q_a \]
\(\chi PT\)

- For masses near the chiral limit, chiral perturbation theory \((\chi PT)\) can be used to make predictions for the forms of matrix elements.
- The leading order chiral Lagrangian is written in terms of

\[
\Sigma = \exp \left[ \frac{2i\phi^a \lambda^a}{f} \right]
\]

where the \(\phi^a\) are the real pseudo-scalar meson fields, and is given by

\[
\mathcal{L}_{LO} = \frac{f^2}{8} Tr[\partial_\mu \Sigma \partial^{\mu} \Sigma] + \frac{f^2 B_0}{4} Tr[\chi^\dagger \Sigma + \Sigma^\dagger \chi]
\]

where \(\chi = diag(m_u, m_d, m_s)\) and

\[
B_0 = \frac{m_{\pi^+}^2}{m_u + m_d} = \frac{m_{K^+}^2}{m_u + m_s} = \frac{m_{K^0}^2}{m_d + m_s}
\]
\[ \chi PT \]

- Matrix elements of the parts of the weak operators that transform in a definite way under SU(3) and isospin are calculated by forming out of the $\Sigma$ field all possible operators that transform in the given way at a given order. A linear combination of these with arbitrary coefficients, called low energy constants (LECs) is taken to represent the weak operator in question.

- At next to leading order and with 0 momentum in the initial and final states, the matrix elements in $\chi PT$ will depend on meson masses squared and to the fourth power, the LECs, and the other parameters in the Lagrangian.
Extraction of LECs

- Since physical pions would require a larger box size than is currently available, the strategy to calculate matrix elements at unphysical kinematics in order to extract LECs from $\chi PT$, and then to use these LECs and $\chi PT$ to calculate matrix elements at physical kinematics.

- In order to extract the necessary LECs for physical $K \rightarrow \pi\pi$ matrix elements given a limited number of ensembles, it is necessary either to resort to partial quenching, in which the masses of the quarks in the fermion determinant are different than the masses of the propagating quarks, or to considering pions with non-zero momentum.
Partial Quenching

- The $24^3 \times 64$ ensembles currently available have sea quark masses $m_u = m_d = 0.005, 0.01$, and $m_s = 0.04$.

- It is necessary to vary both the light quark and sea quark valence masses in general.

- Laiho and Soni (hep-lat 0306035) have treated the case of partially quenched $\chi PT$ at NLO with sea quarks of equal mass. Partially quenched $\chi PT$ formulae at NLO for the case of unequal sea quark masses are in the process of being calculated by Christopher Aubin, Sam Li, and Jack Laiho.
Non-Zero Momenta

- $\chi PT$ for $K \rightarrow \pi\pi$ matrix elements with pions having non-zero momentum has been worked out by Sachrajda et. al. (hep-lat 0208007) and Laiho and Soni (hep-lat 0203106).
- In practice data with non-zero momenta can be very noisy.
- There are some methods for dealing with this, such as antiperiodic, and in general twisted boundary conditions.
I will focus primarily on the partial quenching approach. Non-zero momenta can be incorporated later as a consistency check.

The following set of valence and sea quark masses are expected to be sufficient to determine the needed LECs. (Here \( m_l = m_u = m_d \))

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Calculating Matrix Elements

- It is necessary to contract the existing propagators in order to calculate diagrams relevant to the four quark weak operators.
- There is the added complication for $\Delta I = 1/2$ operators of a vacuum subtraction.
- Once matrix elements are calculated on the lattice, they must be normalized to the lattice using non-perturbative renormalization (NPR). Sam Li has worked on this.
- The LECs can then be calculated from fits to the NLO $\chi PT$ formulae.
Calculations

- Have calculated the matrix element (not normalized) for $(27,1) \Delta I = 3/2$ for sea quark masses $m_s = 0.04, m_u = m_d = 0.005, 0.01$, and valence quark masses

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- The lattices are $24^3 \times 64, L_s = 16$, with an inverse lattice spacing of $a^{-1} = 1.73(2)$ GeV, and are averaged over periodic and antiperiodic boundary conditions in order to double the effective length in the time direction.
Details of Calculation - Diagrams

- For (27,1) $\Delta I = 3/2$ we can calculate

$$\langle \pi^+ \pi^+ | \mathcal{O}'^{(27,1),3/2} | K^+ \rangle$$

and relate it to the physically relevant matrix element via the Wigner-Eckhart theorem

$$\langle \pi^+ \pi^+ | \mathcal{O}'^{(27,1),3/2} | K^+ \rangle = \frac{2\sqrt{2}}{\sqrt{3}} \langle \pi^+ \pi^0 | \mathcal{O}^{(27,1),3/2} | K^+ \rangle \quad (2)$$

where $\mathcal{O}'^{(27,1),3/2} = \bar{s}\gamma_\mu (1 - \gamma^5) d \bar{u}\gamma^\mu (1 - \gamma^5) d$

- $\langle \pi^+ \pi^+ | \mathcal{O}'^{(27,1),3/2} | K^+ \rangle$ has only one diagram (see the following slide)
Details of Calculation - Setup

- Wall sources and sinks at \( t = 5 \) and \( t = 59 \) respectively are used.
- The weak operator is at a point and is moved around in time. Matrix elements are plotted as a function of this time.
Results - Effective Mass Plot $m_s = m_l = 0.04$
Effective Mass Plot $m_s = 0.03, m_l = 0.02$
Effective Mass Plot $m_s = 0.02, m_l = 0.005$
Effective Mass Plot $m_s = 0.005, m_l = 0.001$
Future Plans

- Plan to calculate matrix elements for the $(8,8)$ and $(8,1)$ operators, and the $\Delta I = 1/2$ operators.
- Compare to partially quenched $\chi PT$ when the formulae are available.
- Calculate some matrix elements that have momenta (perhaps the first non-vanishing value) as a consistency check.
- Perform NPR on matrix elements.
- Do the calculations on other lattice sized $(32^3 \times 64)$ in order to do a continuum extrapolation.