Probing the Chiral Limit in 2+1 flavor Domain Wall Fermion QCD

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Outline

1. Introduction
   - Chiral Extrapolations in General
   - Chiral Extrapolations for Domain Wall Fermions

2. Simulations and Results
   - Ensemble Details
   - $m_P, f_P$ and the Chiral Fits
   - Low Energy Constants and Physical $f_\pi, f_K$ and $f_K/f_\pi$

3. Summary and Outlook
Chiral Extrapolations

- Finished 2+1-flavor DWF simulations:
  - \( m_l/m_s \geq 1/5 \) (dynamical)
  - \( m_l/m_s \geq 1/10 \) (partially quenched)

- Use \( SU(3)_L \times SU(3)_R \) chiral perturbation theory to guide the extrapolations.
  - Expand around \( m_u = m_d = m_s = 0 \) limit.
  - Example: \( M_{\pi}^2 \) from \( SU(3) \) ChPT, NLO

\[
M_{\pi}^2 = 2B_0 \overline{m} \left\{ 1 + \frac{32B_0}{f^2} \left[ (2L_8 - L_5)\overline{m} + (2L_6 - L_4)(2\overline{m} + m_s) \right] \right. \\
+ \left. \frac{2B_0}{16\pi^2 f^2} \left[ \overline{m} \log \frac{2B_0 \overline{m}}{\mu^2} - \frac{\overline{m} + 2m_s}{9} \log \frac{2B_0(\overline{m} + 2m_s)}{3\mu^2} \right] \right\}.
\]

- Non-analyticity (chiral logarithms) anticipated at small quark masses
DWF AWI and the Residual Mass

How does the chiral extrapolation change for DWF?

- Continuum QCD axial Ward-Takahashi identity is

\[ \partial_\mu \langle A_\mu^a(x)O(y) \rangle = 2m_f \langle J_5^a(x)O(y) \rangle + i\langle \delta^a O(y) \rangle. \]


\[ \Delta_\mu \langle A_\mu^a(x)O(y) \rangle = 2m_f \langle J_5^a(x)O(y) \rangle + 2\langle J_{5q}^a(x)O(y) \rangle + i\langle \delta^a O(y) \rangle. \]

- The additional term \( J_{5q}^a \) is related to the residual mass \( m_{\text{res}} \)

\[ J_{5q}^a = m_{\text{res}} J_5^a + O(a) \]

- All the lattice quark masses get an additive renormalization:

\[ m_q = m_f + m_{\text{res}} \]

- Instead of using the input quark mass \( m_f \) in the chiral fits, we need to use the total quark mass \( m_q \).
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Symanzik Expansions

How about corrections from higher-dimension operators with $O(a)$?

- To $O(a)$, the Symanzik action for DWF can be written as
  \[
  S = \int d^4x [\overline{\psi}(x)(iD - m_q)\psi(x)] + ae^{-\alpha L_s} c_{dwf} \overline{\psi}(x)\sigma^{\mu\nu} F_{\mu\nu} \psi(x)
  \]
  where $m_q = m_f + m_{res}$.

- Similar to Wilson fermions, (Rupak & Shoresh, PRD 66:054503,2002), but the $O(a)$ lattice artefact is exponentially small for domain wall fermions
  \[
  ae^{-\alpha L_s} c_{dwf} \sim m_{res}(a\Lambda_{QCD})^2 \sim 0.1m_{res},
  \]
  with the assumption that $a \sim 0.1$ fm, and $\Lambda_{QCD} \sim 500$ MeV $\implies O(a)$ term is NLO correction

- Our definition for $m_{res}$ guarantees $M_\pi^2$ vanishes at $m_f + m_{res} = 0 \Rightarrow$ “PCAC” quark mass
  S.Sharpe, arXiv:0706.0218 [hep-lat]
  \[
  m_{res} = \frac{\langle \sum_x J_{5q}^a (x, t) \pi^a(0) \rangle}{\langle \sum_x J_{5}^a (x, t) \pi^a(0) \rangle}
  \]

- Contributions of all orders in $a$ have been taken into account in $m_{res}$
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S.Sharpe, arXiv:0706.0218 [hep-lat]

$$m_{\text{res}} = \frac{\langle \sum_{\vec{x}} J_{5q}^a(\vec{x}, t)\pi^a(0) \rangle}{\langle \sum_{\vec{x}} J_{5}^a(\vec{x}, t)\pi^a(0) \rangle}$$

- Contributions of all orders in $a$ have been taken into account in $m_{\text{res}}$
Simulation Overview

- Used the Iwasaki gauge action

\[ S_g[U] = -\frac{\beta}{3} \left[ (1 - 8c_1) \sum_{x;\mu<\nu} P[U]_{x,\mu\nu} + c_1 \sum_{x;\mu\neq\nu} R[U]_{x,\mu\nu} \right] \]

with \( c_1 = -0.331 \), chosen to balance between a small \( m_{\text{res}} \) and fast topology tunneling

- **RBC and UKQCD 2+1 flavor DWF ensembles**

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( L^3 \times T )</th>
<th>( L_s )</th>
<th>( m_\text{input}^{\text{input}} / m_s^{\text{input}} )</th>
<th>( m_\text{tot}^{\text{tot}} / m_s^{\text{phys}} )</th>
<th># MD Time Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.13</td>
<td>( 16^3 \times 32 )</td>
<td>16</td>
<td>0.01/0.04</td>
<td>0.33</td>
<td>4015</td>
</tr>
<tr>
<td>2.13</td>
<td>( 16^3 \times 32 )</td>
<td>16</td>
<td>0.02/0.04</td>
<td>0.59</td>
<td>4045</td>
</tr>
<tr>
<td>2.13</td>
<td>( 16^3 \times 32 )</td>
<td>16</td>
<td>0.03/0.04</td>
<td>0.85</td>
<td>4020+3580</td>
</tr>
<tr>
<td>2.13</td>
<td>( 24^3 \times 64 )</td>
<td>16</td>
<td>0.005/0.04</td>
<td>0.21</td>
<td>4500</td>
</tr>
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<td>2.13</td>
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<td>0.85</td>
<td>2813</td>
</tr>
</tbody>
</table>

Will focus on the two lightest \( 24^3 \times 64 \) ensembles in this talk.
Partially Quenched Measurements

- Use a set of different valence quark masses at a fixed $m_{sea}$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int [dU] \det[D^\dagger(m_{sea})D(m_{sea})]O(m_1, m_2)e^{-S_g[U]}$$

- More data points to constrain the chiral fits
- Partially quenched measurement parameters

<table>
<thead>
<tr>
<th>$m_l/m_s$</th>
<th>$am_{1,2}$</th>
<th># measurements [sources]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005/0.04</td>
<td>(0.001 0.005) 0.01 0.02 0.03 0.04</td>
<td>704 <a href="ND">2</a> 90 [2] (ND)</td>
</tr>
<tr>
<td>0.01/0.04</td>
<td>(0.001 0.005) 0.01 0.02 0.03 0.04</td>
<td>710 <a href="ND">2</a> 45 [2] (D)</td>
</tr>
<tr>
<td>0.02/0.04</td>
<td>(0.001 0.005) 0.01 0.02 0.03 0.04</td>
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</tr>
<tr>
<td>0.03/0.04</td>
<td>(0.001 0.005) 0.01 0.02 0.03 0.04</td>
<td></td>
</tr>
</tbody>
</table>

ND: all the non-degenerate combinations
D: degenerate mesons only
The Residual Mass

Recall:

\[ J_{5q}^a = m_{\text{res}} J_{5}^a + \mathcal{O}(a) \]

- We compute the ratio \( R(t) \) on the lattice

\[ R(t) = \frac{\langle \sum_{\vec{x}} J_{5q}^a(\vec{x}, t) \pi^a(0) \rangle}{\langle \sum_{\vec{x}} J_5^a(\vec{x}, t) \pi^a(0) \rangle} \]

- At small \( t \)'s, the correlation function may overlap with unphysical states. Only when \( t \) is sufficiently large does \( R(t) \) represent the residual mass.

- Fit \( R(t) \) where plateaux are reached to a constant to get \( m_{\text{res}} \)
The Residual Mass

- $J_{5q}$ and $J_5$ may depend on the quark mass differently, so $m_{\text{res}}$ displays some quark mass dependence.

- Define the residual mass in the limit of $m_f \to 0$:
  \[ am_{\text{res}} \approx 0.0031 \]

- All the quark masses in the simulation get an additional mass shift:
  \[ m_q = m_f + m_{\text{res}} \]
Determine the Lattice Scale from $\Omega^-$ Mass

Advantages of $\Omega^-$ ($J^P = \frac{3}{2}^+$, sss) input

also in Toussaint and Davies, NPB (Proc. Suppl.) 140 (2005) 234

- Statistically well-determined on the lattice
  → better statistical accuracy for $1/a$

- Free of light quark mass chiral logs
  → simple chiral extrapolations

- Stable particle in the QCD sector
  → no complications with decays

Representative effective masses
Determine the Lattice Scale from \( \Omega^- \) Mass

One complication: need to interpolate to the physical strange quark mass \( m_s^{phys} \) from the physical ratio of \( m_K^2/m_{\Omega^-}^2 \)

\[
m_s = \frac{m_K}{m_{\Omega^-}}
\]

Other determinations of \( 1/a \):

- From \( m_\rho \): \( \approx 1.62(5) \) GeV
- From \( r_0 = 0.5 \) fm: \( \approx 1.63(2) \) GeV
- Both consistent with \( 16^3 \times 32 \) simulations (hep-lat/0701013), but systematically low

\[
1/a = 1.73(2) \text{ GeV}
\]
Pseudoscalar Meson Mass $m_P$

- Simultaneous fit to multiple correlators for reduced systematic and statistical errors
- One common ground-state mass, and one amplitude for each correlator included

\[ C^{LW}_{AA}(t) = \langle A^L(t)A^W(0) \rangle \rightarrow A^{LW}_{AA} [\exp(-m_P t) + \exp(-m_P (T - t))] , \]
\[ C^{LW}_{PP}(t) = \langle P^L(t)P^W(0) \rangle \rightarrow A^{LW}_{PP} [\exp(-m_P t) + \exp(-m_P (T - t))] , \]
\[ C^{LW}_{AP}(t) = \langle A^L(t)P^W(0) \rangle \rightarrow A^{LW}_{AP} [\exp(-m_P t) - \exp(-m_P (T - t))] , \]
\[ C^{WW}_{PP}(t) = \langle P^W(t)P^W(0) \rangle \rightarrow A^{WW}_{PP} [\exp(-m_P t) + \exp(-m_P (T - t))] , \]
\[ C^{WW}_{AP}(t) = \langle A^W(t)P^W(0) \rangle \rightarrow A^{WW}_{AP} [\exp(-m_P t) - \exp(-m_P (T - t))] . \]

Superscripts: W - Coulomb gauge fixed wall source; L - local source

Subscripts: A - axialvector operator, P - pseudoscalar operator

- Matrix elements needed in the $f_P$ calculation can be obtained from appropriate ratios
Pseudoscalar Meson Decay Constant $f_P$

$$f_P = \frac{Z_A |\langle 0|A|\pi \rangle|}{m_P} \quad \text{(def.)} \quad \text{or} \quad f_P = \frac{2(m_f + m_{\text{res}})}{m_P} |\langle 0|P|\pi \rangle| \quad \text{(AWI)}$$

Five ways to determine $f_P$ based on different combinations of $\langle 0|P|\pi \rangle$ and $\langle 0|A|\pi \rangle$

Not all independent, choose the one with best systematic and statistical error
Pseudoscalar Meson Decay Constant $f_P$

Methods with $m_{res}$ involved differ $\Rightarrow O(a)$ error?

Choose III in the final analysis (use only the definition, systematically clean and has the smallest error bar)

$$\frac{am_l^{sea}}{am_s^{sea}} = 0.005/0.04$$
Chiral Fits Revisited

- **Small volume data**: to check if NLO ChPT is a valid description in the range of quark masses we are working with

\[
\frac{aM_{PS}^2}{m_{val} + m_{res}}
\]

Pseudoscalar meson masses: 400 - 620 MeV
Simultaneous NLO fit **fails**!
⇒ Masses too heavy; Higher-order corrections non-negligible
**Large volume data** allows for simulations with lighter quark masses without introducing large finite volume effects.

Decreasing the masses included improves the fit quality.

\[ m_{\text{avg}} = \frac{(m_1 + m_2)}{2} \in [0.001, 0.02], \quad m_{\text{sea}} \in [0.005, 0.01] \Rightarrow m_P < 550 \text{ MeV} \]
Large volume data allows for simulations with lighter quark masses without introducing large finite volume effects.

Decreasing the masses included improves the fit quality.

\[ m_{\text{avg}} = \frac{m_1 + m_2}{2} \in [0.001, 0.015], \quad m_{\text{sea}} \in [0.005, 0.01] \Rightarrow m_P < 490 \text{ MeV} \]
Fits Using NLO $SU(3)$ PQChPT

- **Large volume data** allows for simulations with lighter quark masses without introducing large finite volume effects.
- Decreasing the masses included improves the fit quality.
- $m_{\text{avg}} = (m_1 + m_2)/2 \in [0.001, 0.01]$, $m_{\text{sea}} \in [0.005, 0.01] \Rightarrow m_P < 420$ MeV.

---

**Graphs:**

- **Graph 1:** Shows the relationship between $a_P$ and $am_{\text{avg}}$ for different values of $(am_P, am_{\text{res}})$. The graphs include data points and fits for $0.005/0.04$ and $0.01/0.04$.
- **Graph 2:** Demonstrates the same relationship as the first graph, but with an additional parameter $(am_{\text{avg}})^2/(am_{\text{avg}} + am_{\text{res}})$. The data points are clustered around the fitted curves for $0.005/0.04$ and $0.01/0.04$. 
Low Energy Constants

- With $m_P < 420$ MeV, lattice data can be described by NLO $SU(3)$ PQChPT reasonably well.
- From which we obtained the following LECs:

<table>
<thead>
<tr>
<th>$\Lambda_\chi$</th>
<th>$aB_0Z_m$</th>
<th>$a f_0$</th>
<th>$10^4 (2L_8 - L_5)$</th>
<th>$10^4 (2L_6 - L_4)$</th>
<th>$10^4 L_4$</th>
<th>$10^4 L_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>770 MeV</td>
<td>2.35(16)</td>
<td>0.0541(40)</td>
<td>2.47(45)</td>
<td>-0.02(42)</td>
<td>1.36(80)</td>
<td>8.62(99)</td>
</tr>
<tr>
<td>1 GeV</td>
<td></td>
<td></td>
<td>5.23(45)</td>
<td>-0.48(42)</td>
<td>-0.71(80)</td>
<td>2.42(99)</td>
</tr>
</tbody>
</table>

- Also determined $\bar{a}m = 0.00139(7)$ from $m_P^2$, and $f_\pi = 124.7(30)$ MeV.

- Problem: Strange quark mass outside of the range of NLO ChPT
- How are we going to determine $f_K$?
  - Adding analytic NNLO terms does make the fits go through the data points up to the strange quark mass, but light quark limit is also changed.
  - Alternatives?
Simulations and Results

Low Energy Constants and Physical $f_\pi, f_K$ and $f_K/f_\pi$

⇒ using $SU(2)_L \times SU(2)_R$ NLO PQChPT

- Expand around $m_u = m_d = 0$ limit, strange quark mass irrelevant for light quark limit
- For quantities involving kaons, use $SU(2)_L \times SU(2)_R$ ChPT in the presence of a heavy $m_s \Rightarrow$ heavy meson ChPT, $SU(2)$ chiral limit unaffected
- Fit several valence $m_s$ and then interpolate to the physical strange quark mass

\[ a_{m_s}^{val} = 0.04 \]
Final Results

- **Preliminary** final results from $SU(2)$ NLO chiral fits

  \[
  f_\pi = 124.2(35) \text{ MeV} \quad [130.7(41) \text{ MeV}]
  \]

  \[
  f_K = 150.0(36) \text{ MeV} \quad [159.8(15) \text{ MeV}]
  \]

  \[
  f_K/f_\pi = 1.208(14) \quad [1.223(12)]
  \]

- It’s also interesting to compare the LECs from $SU(3)$ and $SU(2)$ fits

<table>
<thead>
<tr>
<th>LEC</th>
<th>$SU(3)$</th>
<th>$SU(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$aB_0Z_m$</td>
<td>2.453(75)</td>
<td>2.414(61)</td>
</tr>
<tr>
<td>$af_0$</td>
<td>0.0662(17)</td>
<td>0.665(21)</td>
</tr>
<tr>
<td>$\bar{l}_3$</td>
<td>2.87(28)</td>
<td>3.13(33)</td>
</tr>
<tr>
<td>$\bar{l}_4$</td>
<td>4.09(5)</td>
<td>4.42(14)</td>
</tr>
</tbody>
</table>

- For more details of the $SU(2)$ chiral fits and quark masses, see talk by Enno Scholz in H10 session @ 15:00 Today.
NLO ChPT predictions $a m_1 = a m_2 = a m_1^{\text{sea}} = 0.01, m_\pi L \approx 3.9$: $\sim -1\%$ FSE on $f_P$

Colangelo et al. 2005 (resummed Luscher + NNLO) predicts $\sim -1.5\%$ FSE

Our data shows a difference of $\sim 3.5(9)\%$ between $L = 16$ and $L = 24$ results

Light valence point with $a m_1,2 = 0.001$ on the $L = 24$ lattices may also suffer from visible finite volume effects

Further investigations needed...
Summary

- NLO $SU(3)$ ChPT is possibly accurate up to $m_P \approx 400$ MeV;
- Within the chiral regime, $SU(3)$ and $SU(2)$ give similar level of accuracy in probing the light quark chiral limit;
- $SU(2)$ heavy meson ChPT provides a useful tool for extrapolations/interpolations up to the strange quark mass.

Outlook
- “Heavy meson” ChPT works with the assumption: $\frac{m_\pi^2}{m_K^2} \ll 1$. Lighter would be better.
- To get a more accurate determination of $f_K$, another dynamical $m_s$ will be needed.
- Finer lattice spacing ($\sim 2.1$ GeV) and lighter quark masses ($\sim m_s/7$) under way:
  - moving inside chiral regime, and continuum extrapolation possible.