On Majorana fermions on the lattice

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Regensburg, July 30th, 2007
1. Chiral fermions on the lattice
   - Nielsen-Ninomiya no–go theorem

2. Majorana fermions on the lattice
   - Majorana no-go
   - Majorana lattice fermions

3. summary
1. Chiral fermions on the lattice
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2. Majorana fermions on the lattice
   - Majorana no-go
   - Majorana lattice fermions

3. Summary
Nielsen-Ninomiya no–go theorem

Dirac action

\[ S = \sum_{x,y \in \Lambda} \bar{\psi}(x) D(x - y) \psi(y) \]

- compatibility of chiral projections, \( P_{\psi,\bar{\psi}}^2 = P_{\psi,\bar{\psi}} \),

\[ D P_{\psi} = P_{\bar{\psi}} D \]
Dirac action

\[ S = \sum_{x,y \in \Lambda} \bar{\psi}(x) D(x - y) \psi(y) \]

- compatibility of chiral projections, \( P_{\psi,\bar{\psi}}^2 = P_{\psi,\bar{\psi}} \)

\[ D P_{\psi} = P_{\bar{\psi}} D \]

- example: Ginsparg-Wilson fermions in 4d
  - GW-relation: \( \gamma_5 D - D \gamma_5 = aD \gamma_5 D \)
  - projection operators

\[ P_{\psi} = \frac{1}{2} (1 - \gamma_5), \quad P_{\bar{\psi}} = \frac{1}{2} (1 + \gamma_5) \]

with \( \gamma_5^2 = 1 \) and \( \gamma_{\psi} = \gamma_5(1 - aD), \quad \gamma_{\bar{\psi}} = \gamma_5. \)
Nielsen-Ninomiya no–go theorem

Dirac action

\[ S = \sum_{x, y \in \Lambda} \bar{\psi}(x) D(x - y) \psi(y) \]

- compatibility of chiral projections: \( D P_\psi = P_{\bar{\psi}} D \)
- locality

\[ |D_{ij}(x)|, |P_{\bar{\psi}}_{ij}(x)|, |P_{\psi}_{ij}(x)| < c e^{-|x|/\lambda} \]
Nielsen-Ninomiya no–go theorem

Dirac action

$$S = \sum_{x,y \in \Lambda} \bar{\psi}(x) D(x - y) \psi(y)$$

- compatibility of chiral projections: $$D P_\psi = P_{\bar{\psi}} D$$
- locality
- spin-$\frac{1}{2}$ zeros

$$D(k)^{-1} = \frac{(k_\mu - k_\mu^{(i)})}{|k - k^{(i)}|^2} \sum_{\mu} (i)^\dagger + \text{finite}$$
Nielsen-Ninomiya no–go theorem

Dirac action

\[ S = \sum_{x,y \in \Lambda} \bar{\psi}(x)D(x-y)\psi(y) \]

- compatibility of chiral projections: \( DP_{\psi} = P_{\bar{\psi}}D \)
- locality
- spin-\( \frac{1}{2} \) zeros

Then the total chirality \( \chi \) is given by

\[ \chi = n[P_{\psi}] - n[P_{\bar{\psi}}] \]

with

\[ n[P] \equiv \frac{1}{l!} \left( \frac{i}{2\pi} \right)^l \int_{T^{2l}} \text{tr} \ P(dP)^{2l} \in \mathbb{Z} \]
Nielsen-Ninomiya no–go theorem

Dirac action

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Then the total chirality \( \chi \) is given by

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\( \psi \) and \( \bar{\psi} \) live in topologically different spaces!

'go-go theorem'
Nielsen-Ninomiya no–go theorem

Dirac action

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On Majorana fermions on the lattice
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

Majorana reduction

\[ \psi = \chi + i \eta, \quad \bar{\psi} = \chi^T C + i \eta^T C \]

with charge conjugation \( C \).
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{\mathbf{x},\mathbf{y} \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \( \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T C + i\eta^T C \)

- Properties of \( C \), continuum

\[
\begin{align*}
C \gamma_\mu C^{-1} & = -\gamma^T_\mu \\
C \gamma_5 C^{-1} & = \gamma^T_5 \\
C^\dagger C & = 1 \\
C^T & = -C.
\end{align*}
\]
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \[ \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T C + i\eta^T C \]
- Properties of \( C \)
- Skew symmetry of \( D \)

\[ (C D)^T = -C D \]
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \( \psi = \chi + i\eta \), \( \bar{\psi} = \chi^T C + i\eta^T C \)
- \( C\gamma_5 C^{-1} = \gamma_5^T \) and skew symmetry: \( (C D)^T = -C D \)
- Chiral invariance \( \psi \rightarrow \gamma \psi \), \( \bar{\psi} \rightarrow \bar{\psi} \gamma \bar{\psi} \)
Majorana no-go

Majorana action

\[
S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi
\]

- Majorana reduction: \( \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T C + i\eta^T C \)
- \( C\gamma_5 C^{-1} = \gamma_5^T \) and skew symmetry: \( (C D)^T = -C D \)
- Chiral invariance \( \psi \rightarrow \gamma \psi \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \gamma \bar{\psi} \)
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \( \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T C + i\eta^T C \)
- \( C \gamma_\bar{\psi} C^{-1} = \gamma_\psi^T \) and skew symmetry: \( (C \, D)^T = -C \, D \)
- Chiral invariance \( \psi \rightarrow \gamma_\psi \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \gamma_\bar{\psi} \).
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \( \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T C + i\eta^T C \)
- \( C\gamma_{\bar{\psi}}C^{-1} = \gamma^T_{\psi} \) and skew symmetry: \( (C D)^T = -C D \)
- Chiral invariance: \( \psi \rightarrow \gamma_{\psi} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \gamma_{\bar{\psi}} \)

\( \psi \) and \( \bar{\psi} \) live in topologically different spaces!

\[ C\gamma_{\bar{\psi}}C^{-1} \neq \gamma^T_{\psi} \]

for smooth invertible \( C \)'s! possible solution \( C = C_{\text{cont}}(1 - \frac{1}{2} aD) \) vanishes at doublers
Majorana no-go

Majorana action

\[ S = \frac{1}{2} \sum_{x,y \in \Lambda} \bar{\psi} D \psi \]

- Majorana reduction: \( \psi = \chi + i\eta, \quad \bar{\psi} = \chi^T \gamma \bar{\psi} + i\eta^T \gamma \bar{\psi} \)
- \( C \gamma \bar{\psi} C^{-1} = \gamma^T \psi \) and skew symmetry: \( (C D)^T = -C D \)
- Chiral invariance \( \psi \rightarrow \gamma \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \gamma \)

\( \psi \) and \( \bar{\psi} \) live in topologically different spaces!

Majorana action

\[ S = \sum_{x,y \in \Lambda} \eta_{\bar{\psi}}^T C D \chi_{\psi} \]

• Majorana reduction

\[ \chi_{\psi} = P_{\psi} \chi, \quad \eta_{\bar{\psi}}^T C = \eta_{\bar{\psi}}^T C \bar{P}_{\bar{\psi}} \]
Majorana lattice fermions

Majorana action

\[ S = \sum_{x,y \in \Lambda} \eta_{\bar{\psi}}^T C D \chi_{\psi} \]

- Majorana reduction

\[ \chi_{\psi} = P_{\psi} \chi, \quad \eta_{\bar{\psi}}^T = \eta_{\bar{\psi}}^T \hat{P}_{\bar{\psi}} \]

with \( \hat{P}_{\bar{\psi}} = C P_{\bar{\psi}} C^{-1} \), \( \hat{\gamma}_{\bar{\psi}} = C \gamma_{\bar{\psi}} C^{-1} \).
Majorana lattice fermions

Majorana action

$$ S = \sum_{x,y \in \Lambda} \eta_{\psi}^T C D \chi_{\psi} $$

- Majorana reduction $\chi_{\psi} = P_{\psi} \chi$, $\eta_{\psi}^T = \eta_{\psi}^T \hat{P}_{\psi}$ with $\hat{P}_{\psi} = C P_{\bar{\psi}} C^{-1}$.

- example: Ginsparg-Wilson fermions in 4d

  - projection operators

    $$ P_{\psi} = \frac{1}{2} (1 - \gamma_{\psi}) , \quad P_{\bar{\psi}} = \frac{1}{2} (1 + \gamma_{\bar{\psi}}) , $$

    with $\gamma_{\psi, \bar{\psi}} = 1$ and $\gamma_{\psi} = \gamma_5 (1 - aD)$, $\gamma_{\bar{\psi}} = \gamma_5$.

  - $\hat{\gamma}_{\psi} = (1 - aD) \gamma_5$, $\hat{\gamma}_{\bar{\psi}} = \gamma_5$
Majorana lattice fermions

Majorana action

\[ S = \sum_{x,y \in \Lambda} \eta_{\psi}^T C D \chi_{\psi} \]

- Majorana reduction \( \chi_{\psi} = P_{\psi} \chi \), \( \eta_{\psi}^T = \eta_{\psi}^T \hat{P}_{\psi} \) with \( \hat{P}_{\psi} = C P_{\bar{\psi}} C^{-1} \).

- skew symmetry : \( (C D)^T = -C D \)
Majorana lattice fermions

Majorana action

\[ S = \sum_{x,y \in \Lambda} \eta^T_{\bar{\psi}} C D \chi_\psi \]

- Majorana reduction \( \chi_\psi = P_\psi \chi \), \( \eta^T_{\bar{\psi}} = \eta^T_{\bar{\psi}} \hat{P}_\psi \) with \( \hat{P}_\psi = C P_{\bar{\psi}} C^{-1} \).
- Skew symmetry: \( (C D)^T = -C D \)
- Chiral invariance trivial

\[ \chi_\psi \to \gamma_\psi \chi_R, \quad \eta^T_{\bar{\psi}} \to \eta^T_{\bar{\psi}} \gamma_{\bar{\psi}} \]
Majorana lattice fermions

Majorana action

$$S = \sum_{x,y \in \Lambda} \eta^T_{\psi} C D \chi_{\psi}$$

- Majorana reduction \( \chi_{\psi} = P_{\psi} \chi \), \( \eta^T_{\psi} = \eta^T_{\psi} \hat{P}_{\psi} \) with \( \hat{P}_{\psi} = C P_{\psi} C^{-1} \).
- Skew symmetry : \( (C D)^T = -C D \)
- Chiral invariance trivial \( \chi_{\psi} \rightarrow \gamma_{\psi} \chi_R \), \( \eta^T_{\psi} \rightarrow \eta^T_{\psi} \gamma_{\psi} \)

It follows that

$$S = \sum_{x,y \in \Lambda} \eta^T_{\psi} C D \chi_{\psi} = \sum_{x,y \in \Lambda} \chi^T_{\psi} C D \eta_{\psi}$$
Majorana lattice fermions

Majorana action

\[ S = \sum_{x,y \in \Lambda} \eta_{\bar{\psi}}^T C D \chi_{\psi} \]

- Majorana reduction \( \chi_{\psi} = P_{\psi} \chi \), \( \eta_{\bar{\psi}}^T = \eta_{\bar{\psi}}^T \hat{P}_{\bar{\psi}} \) with \( \hat{P}_{\bar{\psi}} = C P_{\bar{\psi}} C^{-1} \).

- Skew symmetry: \( (C D)^T = -C D \)

- Chiral invariance trivial \( \chi_{\psi} \rightarrow \gamma_{\psi} \chi_R \), \( \eta_{\bar{\psi}}^T \rightarrow \eta_{\bar{\psi}}^T \gamma_{\bar{\psi}} \)

- Yukawa action chirally invariant with: \( \varphi \rightarrow 2 \varphi \)

\[ S_Y = g \sum_{x,y \in \Lambda} \left( \chi_{\bar{\psi}}^T C \varphi \chi_{\bar{\psi}} + \eta_{\psi}^T C \varphi^\dagger \eta_{\psi} \right) \]
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3. summary
summary

- topological obstruction for chiral fermions on the lattice
- topological obstruction for Majorana fermions on the lattice
- construction of free lattice Majorana fermions
- Yukawa interaction