The distributions of individual Dirac eigenvalues for QCD at non-zero chemical potential: RMT predictions and Lattice results

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Plan of the talk

- Motivation and open questions
- Gap probabilities and individual eigenvalue distributions for complex eigenvalues
- Results from Chiral Random Matrix Theory (chRMT)
- Lattice simulations
- The comparison
- Conclusions
Motivation and open questions

General

• Studies of the low-energy Dirac spectrum provide important information on the chiral structure of QCD

• In the ergodic regime of QCD, Lattice results can be compared to analytical predictions of the chiral Random Matrix Theory (chRMT)

• At zero quark (barion) density ($\mu = 0$) lattice results agree very well with chRMT (Edwards, Heller, Kisikis & Narayanan’99)

• $\mu \neq 0$: problems and recent developments
  
  – Sign problem in unquenched Lattice simulations
  
  – Dirac operator non-hermitian, and eigenvalues complex in general
  
  – Recent developments in non-hermitian chRMT (Akemann’02, Splittorff & Verbaarschot’03, Osborn’03), all spectral correlations are known explicitly.
  
  – Generalization of the overlap formalism to $\mu \neq 0$ (Bloch & Wettig’06)
  
  – First comparison of Lattice results and chRMT for both trivial and non-trivial topology (essentially, on the level of spectral density) - good agreement (Bloch & Wettig’06)
Individual eigenvalue distributions

• Possible probes
  – Smallest eigenvalue(s)
  – Microscopic spectral density
  – Number variance

• Individual eigenvalue distributions $p_k(\lambda)$ are a better probe than the microscopic spectral density:
  – Localized distributions, more sensitive
  – Natural object to compute from the Lattice data
  – Potentially can provide more information

• For $\mu = 0$, all $p_k(\lambda)$ are known analytically in chRMT (Damgaard, Nishigaki & Wettig’98, Damgaard & Nishigaki’01), in excellent agreement with the Lattice results (Edwards et al, ’99)

• Questions to answer for $\mu \neq 0$:
  – How to order complex eigenvalues (i.e., what means "first", "second", etc)?
  – Analytical predictions from chRMT?

• The only known RMT result of this kind is for Ginibre ensemble (Grobe, Haake & Sommers’88) - different symmetry class than chRMT
Gap probabilities and eigenvalue distributions for complex eigenvalues

1. Gap probabilities: direct generalization from 1D

- $k$ - point correlation functions for $N$ complex eigenvalues:
  \[
  \rho_k(\lambda_1, \ldots, \lambda_k) = \frac{N!}{(N-k)!} \int_{C^{N-k}} d^2 \lambda_{k+1} \ldots d^2 \lambda_N \mathcal{P}_N(\lambda_1, \ldots, \lambda_N).
  \]

- Ordering problem: how to order complex eigenvalues?

- The gap probability $E_k[J]$ for a closed region $J$:
  \[
  E_k[J] = \frac{N!}{(N-k)!} \int_{J^k} d^2 \lambda_1 \ldots d^2 \lambda_k \times \\
  \times \int_{(C/J)^{N-k}} d^2 \lambda_{k+1} \ldots d^2 \lambda_N \mathcal{P}_N(\lambda_1, \ldots, \lambda_N).
  \]

- In the complex case $E_k[J]$ are functionals of the 2D domain $J$.

- Still, $E_k[J]$ and $\rho_l(\lambda_1, \ldots, \lambda_l)$ relate similarly to in the real case
  \[
  E_k[J] = \sum_{l=0}^{N-k} (-1)^l \frac{1}{l!} \int_{J^{k+l}} d^2 \lambda_1 \ldots d^2 \lambda_{k+l} \rho_{k+l}(\lambda_1, \ldots, \lambda_{k+l}).
  \]
2. Eigenvalue distributions and gap probabilities

- We define $k$-th eigenvalue distribution with respect to the domain $J$:

$$ p_k^J(\tau) = k \binom{N}{k} \int_{J^{(k-1)}} d^2 \lambda_1 ... d^2 \lambda_{k-1} \int_{(C/J)^{N-k}} d^2 \lambda_{k+1} ... d^2 \lambda_N $$

$$ \times \mathcal{P}_N(\lambda_1, ..., \lambda_{k-1}, x(\tau) + iy(\tau), \lambda_{k+1}, ..., \lambda_N), $$

where $\{x(\tau), y(\tau)\}$ parametrize the boundary contour $\partial J$.

- Notice that for any functional $F^J[f]$ of the form

$$ F^J[f] = \int_J dxdy f(x, y), $$

it is true that

$$ \frac{\delta F^J}{\delta ((\partial J)(\tau))} = f(x(\tau), y(\tau)) $$

where the variation is with respect to the contour $\partial J$. 
• Using this, we get in analogy with 1D case:

\[
\frac{\delta E_k[J]}{\delta((\partial J)(\tau))} = k! [p_k^J(\tau) - p_{k+1}^J(\tau)]
\]

• In particular, for the first eigenvalue distribution:

\[
p_1(\tau) = \rho_1(x(\tau), y(\tau)) - \int_J d^2 \lambda \rho_2(\lambda, x(\tau) + iy(\tau)) + \\
+ \frac{1}{2} \int_{J \times J} d^2 \lambda_1 d^2 \lambda_2 \rho_3(\lambda_1, \lambda_2, x(\tau) + iy(\tau)) - \ldots
\]

• Ordering defined only with respect to a given contour \(J\). So, \(J\) becomes a part of the definition of \(p_k\).

• We want to compare \(p_k\)-s defined on different contours. This can be done for any family of mutually non-intersecting contours which span some domain in \(C\).

• For a general family of contours, \(p_k\)-s have to be slightly redefined to be properly normalized.

• Certain 1-parametric families of contours such as circles and ellipses are simpler than others.

• A practical programme for computations.
Results from chiral RMT

1. Non-Hermitian Chiral 2-matrix model

- The specific matrix model we use is the 2-matrix model for QCD with a barion chemical potential $\mu$, defined in Osborn’04:

$$Z_{\nu} = \mathcal{N} \int dA dB w_G(A)w_G(B) \prod_{i=1}^{N_f} \det(D + m_i),$$

$$D = \begin{pmatrix} 0 & iA + \mu B \\ iA^\dagger + \mu B^\dagger & 0 \end{pmatrix}, \quad w_G(A) = \exp\{-N Tr[A^2]\}$$

where $\nu$ is the (integer) topological charge, $D$ is the Dirac operator, and $A$ and $B$ are complex $(N + \nu) \times N$ matrices.

- The resulting Dirac eigenvalue representation has a non-gaussian weight:

$$Z_{\nu} = \mathcal{N}' \prod_{j=1}^{N_f} m_j^\nu \int \prod_{k=1}^{N} \left[ w(z_k) \prod_{j=1}^{N_f} (z_k^2 + m_j^2) dz_k \right] |\Delta(z^2)|^2,$$

$$w(z) = |z|^{2\nu+2} \exp\left\{ \frac{N(1 - \mu^2)}{4\mu^2} (z^2 + \bar{z}^2) \right\} K_{\nu}\left( \frac{N(1 + \mu^2)}{2\mu^2}|z|^2 \right)$$

- The polynomials orthogonal on $\mathcal{C}$ with this weight are Laguerres:

$$P_k(z) = \left( \frac{\mu^2 - 1}{N} \right)^k k! L_k^\nu\left( \frac{Nz^2}{1 - \mu^2} \right)$$
• The exact spectral microscopic kernel of the model reads for $N_f = 0$ (Splittorff & Verbaarschot’03, Osborn’04)

$$K_s(x, y) = \frac{|xy|^{\nu+1}}{2\pi \alpha^2 (xy^*)^\nu} \sqrt{K_\nu\left(\frac{|x|^2}{2\alpha^2}\right)K_\nu\left(\frac{|y|^2}{2\alpha^2}\right)} \times$$

$$\times e^{\frac{Re(x^2+y^2)}{4\alpha^2}} \int_0^1 e^{-\alpha^2 t} J_\nu(x\sqrt{t})J_\nu(y^*\sqrt{t}).$$

• The microscopic limit is taken as:

$$\rho_S(\xi_1, \ldots, \xi_n) = \lim_{N \to \infty} \frac{1}{N^{2n}} \rho_N\left(\frac{\xi_1}{N}, \ldots, \frac{\xi_n}{N}\right)$$

• The regime of weak non-Hermiticity (Fyodorov, Khoruzhenko, Sommers’97):

$$\alpha^2 = 2N \mu^2 - \text{fixed, } N \to \infty$$

• The $k$-point functions can be expressed as determinants of the kernel (Akemann’02, Osborn’04).

• In the limit $\alpha \to 0$ we have

$$\rho_\nu(z) = \delta(Im(z)) \rho^{ch\text{GUE}}_\nu(Re(z))$$

• In the limit: $\mu = 1, N \to \infty (\alpha \to \infty)$ (strong non-Hermiticity) - rotational invariance:

$$\rho_\nu(z) = \frac{1}{\pi} |z|^2 K_\nu(|z|^2) I_\nu(|z|^2)$$
2. Exact results at strong non-Hermiticity

• For \( N_f = 0 \) the probability that the domain \( J \) is free from the eigenvalues (modulo exact zero modes) is given by

\[
E_0[J] = \frac{1}{Z} \int_{C/J} dz_1 \ldots dz_N \prod_{k=1}^{N} w(z_k) |\Delta(z^2)|^2
\]

• For \( \mu = 1 \), the measure is rotationally invariant:

\[
w(z) \equiv w(|z|) = |z|^{2\nu+2} K_{\nu}(N|z|^2)
\]

• This symmetry allows us to obtain an exact expression:

\[
E_0(R) = \prod_{k=0}^{N} \left\{ \left( R^2 \right)^{2k+\nu+1} K_{\nu+1}(R^2) \over 2^{(2k+\nu)} k!(k+\nu)! \right\} + \right.
\]

\[
+ R^2 \left( K_{\nu+1}(R^2) I_{\nu+2}^{[k-2]}(R^2) + K_{\nu+2}(R^2) I_{\nu+1}^{[k-1]}(R^2) \right)
\]

where we introduced an incomplete Bessel function:

\[
I_{\nu}^{[k]}(x) = \sum_{n=0}^{k} \frac{(x/2)^{2n+\nu}}{n!(n+\nu)!}
\]

• Similar expressions for all \( E_k \)'s can be written. The eigenvalue distributions are then computed with

\[
p_k(R) = - \frac{\partial}{\partial R^2} \sum_{n=0}^{k-1} \frac{E_n(R)}{n!}
\]

• Note that this case corresponds to circular contours \( J \) and we have a well-defined radial ordering.
• First few eigenvalue distributions
3. Fredholm determinant expansion at weak non-Hermiticity

- Weak non-Hermiticity corresponds to finite $\alpha = \mu \sqrt{2N}$.
  For finite volume ($N$ in RMT), $\mu$ is also finite. This is the case on the Lattice.

- Our approach in this case is based on the formula
  \[
  p_1(\tau) = \rho_1(x(\tau), y(\tau)) - \int_J d^2 \lambda \rho_2(\lambda, x(\tau) + iy(\tau)) + \\
  + \frac{1}{2} \int_{J \times J} d^2 \lambda_1 d^2 \lambda_2 \rho_3(\lambda_1, \lambda_2, x(\tau) + iy(\tau)) - ... 
  \]

- In practice we compute it up to the $\rho_3$ term. The convergence is very fast.

- Motivation: this procedure was carried out before for $\mu = 0$ case (Akemann & Damgaard’03), where exact results are known (Damgaard & Nishigaki’01) and the fast convergence was observed.

- For the $\mu \neq 0$ case (2-dimensional), we have to specify the family of contours. We considered ellipses:
  \[
  \text{Re}(z) = \eta \cos \phi , \quad \text{Im}(z) = \frac{a}{b} \eta \sin \phi ,
  \]
  where $a$ and $b$ are fixed for a given contour family, while a given contour is characterized by the "radius" $\eta$, $0 \leq \eta \leq b$. 
• The 1D result:

The distribution of the first eigenvalue \( \rho_1(\lambda) \) for \( \nu = 0 \).

• The 2D result:

One-point density \( \rho_1(\lambda) \) for \( \nu = 0 \) and \( \mu = 0.1 \) (\( \sigma = 0.591 \)).

Computations were performed with Mathematica and DCUHRE routine (Alan Genz et al.) for multidimensional adaptive integration.
Results for $\nu = 0$ and increasing $\mu(\alpha)$

- Weakest non-Hermiticity (corresponds to $\mu = 0.1$ on the Lattice)

One-point density $\rho_1(\lambda)$, for $\nu = 0$ and $\mu = 0.1$ ($\alpha = 0.591$)

The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 0$ and $\mu = 0.1$ ($\alpha = 0.591$)
• Weak non-Hermiticity ($\mu = 0.2$)

One-point density $\rho_1(\lambda)$, for $\nu = 0$ and $\mu = 0.2$ ($\alpha = 1.109$)

The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 0$ and $\mu = 0.2$ ($\alpha = 1.109$)
• **Stronger non-Hermiticity** ($\mu = 0.3$)

One-point density $\rho_1(\lambda)$, for $\nu = 0$ and $\mu = 0.3$ ($\alpha = 1.683$)

The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 0$ and $\mu = 0.3$ ($\alpha = 1.683$)
Results for $\nu = 1$ and increasing $\mu(\alpha)$

- Weakest non-Hermiticity (corresponds to $\mu = 0.1$ on the Lattice)

One-point density $\rho_1(\lambda)$, for $\nu = 1$ and $\mu = 0.1$ ($\alpha = 0.591$)

The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 1$ and $\mu = 0.1$ ($\alpha = 0.591$)
• **Weak non-Hermiticity (\( \mu = 0.2 \))**

One-point density \( \rho_1(\lambda) \), for \( \nu = 1 \) and \( \mu = 0.2 \) (\( \alpha = 1.109 \))

The distribution of the first eigenvalue \( p_1(\lambda) \) for \( \nu = 1 \) and \( \mu = 0.2 \) (\( \alpha = 1.109 \))
- Stronger non-Hermiticity ($\mu = 0.3$)

One-point density $\rho_1(\lambda)$, for $\nu = 1$ and $\mu = 0.3$ ($\alpha = 1.683$)

The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 1$ and $\mu = 0.3$ ($\alpha = 1.683$)
Lattice simulations

- Wilson Dirac operator at $\mu \neq 0$

$$D_W(\mu) = 1 - k \sum_{i=1}^{3} (T_i^+ + T_i^-) - k(e^{\mu T_4^+} + e^{-\mu T_4^-}),$$

$$(T_{\nu}^\pm) = (1 \pm \gamma_\nu)U_{\pm \nu}(x)\delta_{y,x,\pm \nu}, \quad k = \frac{1}{2m_W + 8}$$

- At $\mu \neq 0$ the overlap operator is defined (Bloch, Wettig’06) as

$$D_{OV}(\mu) = 1 + \gamma_5 \epsilon(\gamma_5 D_W(\mu))$$

- Main properties of $D_{OV}$
  - $\gamma_5 D_{OV}(\mu) \gamma_5 = D_{OV}^\dagger (-\mu)$ - no longer $\gamma_5$-Hermitian
  - $D_{OV}(\mu)$ satisfies the Ginsparg - Wilson relation
  - Still have exact lattice chiral symmetry and exact zero modes
  - For an eigenfunction $\psi_{\lambda}$, $\lambda \neq 0, 2$, $\psi_{\lambda'} = \gamma_5 \psi_{\lambda}$ is also an eigenfunction with $\lambda' = \lambda/(\lambda - 1)$
  - For $\lambda = 0, 2$ $\psi_{\lambda}$ have definite chirality

- Simulation details
  - Simulation performed for $\mu = 0.1, 0.2, 0.3, 1.0$
  - For each $\mu$, 7000 - 9000 configurations
  - The parameters were: $\beta = 5.1$, $V = 4^4$, $m_W = -2$
Lattice vs. chRMT - a comparison

• To have a good statistics, we compare the first eigenvalue distribution integrated over the polar angle $\phi$

$$P_1(\eta) = \int_{-\pi}^{\pi} d\phi \ p_1(\eta, \phi)$$

• For lattice data, binning in $\eta$ is performed, according to the definition of ellipses (or other contours we choose)

• To compare to RMT, the $\Sigma$ and $f_\pi$ were determined from a 2-parametric fit to the RMT microscopic spectral density

• Eigenvalues have to be rescaled as $z = \lambda V \Sigma$

• RMT description is valid when $m_\pi, \mu << \frac{1}{L} << \Lambda_{QCD}$, or $|z| << f^2_\pi \sqrt{V}$. 
Weakest non-Hermiticity ($\mu = 0.1$)

- The density (previous work - Bloch & Wettig’06)

\[
\rho(0,y)
\]

- The first eigenvalue
Weak non-Hermiticity ($\mu = 0.2$)

- The density (previous work)
  
  \[ \rho(0,y) \]

- The first eigenvalue
Stronger non-Hermiticity ($\mu = 0.3$)

- The density (previous work)

- The first eigenvalue
Even stronger non-Hermiticity ($\mu = 1.0$)

- The density (previous work)

- The first eigenvalue
Conclusions

• Ordering of complex eigenvalues is possible with respect to a given family of mutually non-intersecting contours in the complex plane

• Known relations between spectral correlation functions and individual eigenvalue distributions can be generalised to complex eigenvalues

• In the case of strong non-Hermiticity, we obtained exact analytical predictions in chRMT

• Fredholm determinant expansion works very well for weak non-Hermiticity, as was expected from the $\mu = 0$ case

• The chRMT predictions are in excellent agreement with the Lattice results