Schrödinger Functional Boundary Conditions for Ginsparg Wilson Quarks

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- Schrödinger functional boundary conditions and Ginsparg Wilson quarks
- Chirally rotated Schrödinger functional
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In some cases SF renormalisation schemes might be the only viable game in town; main goal: apply SF renormalization schemes to Ginsparg-Wilson type quarks

- use cheap regularisation (Wilson quarks, staggered quarks) to obtain the universal running (of coupling, quark masses, composite operators) in the continuum limit.

- use GW regularisation of SF to match SF scheme at some reference scale

⇒ mainly need SF with massless quarks!

What is the problem?

- SF b.c.’s are natural for Wilson quarks:

$$D_W \psi(x_0) = -P_- \psi(x_0 + a) + K \psi(x_0) - P_+ \psi(x_0 - a), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_0).$$

→ obtain the Dirac operator for the SF by just setting the projected fields at the boundaries to zero.
• However, this does not work with GW quarks:
  – this corresponds to choosing SF boundary conditions for the Wilson kernel of Domain Wall/overlap quarks
  ⇒ the Ginsparg-Wilson relation then still holds, despite chiral symmetry violation by the SF boundary conditions!
  ⇒ something’s rotten...

**Proposed solutions:**

• direct orbifold construction *(Taniguchi ’04)*

• “symmetries plus universality” *(Lüscher ’06)*

• here: orbifold construction of chirally rotated SF *(see also Taniguchi ’06)*
Consider isospin doublets $\chi'$ and $\bar{\chi}'$ satisfying homogeneous SF boundary conditions $(P_{\pm} = \frac{1}{2}(1 \pm \gamma_0))$,

\[ P_+\chi'(x)|_{x_0=0} = 0, \quad P_-'(x)|_{x_0=T} = 0, \]
\[ \bar{\chi}'(x)P_-|_{x_0=0} = 0, \quad \bar{\chi}'(x)P_+|_{x_0=T} = 0. \]

perform a chiral field rotation,

\[ \chi' = \exp(i\alpha \gamma_5 \tau^3/2)\chi, \quad \bar{\chi}' = \bar{\chi} \exp(i\alpha \gamma_5 \tau^3/2), \]

the rotated fields satisfy chirally rotated boundary conditions

\[ P_+(\alpha)\chi(x)|_{x_0=0} = 0, \quad P_-(\alpha)\chi(x)|_{x_0=T} = 0, \]
\[ \bar{\chi}(x)\gamma_0P_-(\alpha)|_{x_0=0} = 0, \quad \bar{\chi}(x)\gamma_0P_+(\alpha)|_{x_0=T} = 0, \]

with the projectors

\[ P_{\pm}(\alpha) = \frac{1}{2} \left[ 1 \pm \gamma_0 \exp(i\alpha \gamma_5 \tau^3) \right]. \]
Special cases of $\alpha = 0, \pi/2$:

$$P_{\pm}(0) = P_{\pm}, \quad P_{\pm}(\pi/2) \equiv Q_{\pm} = \frac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

The chiral rotation introduces a mapping between renormalised correlation functions

$$\langle O[\chi, \bar{\chi}] \rangle_{P_{\pm}} = \langle \tilde{O}[\chi, \bar{\chi}] \rangle_{P_{\pm}(\alpha)}$$

with

$$\tilde{O}[\chi, \bar{\chi}] = O\left[\exp(i\alpha\gamma_5\tau^3/2)\chi, \bar{\chi}\exp(i\alpha\gamma_5\tau^3/2)\right],$$

Boundary quark fields are included by replacing

$$\bar{\zeta}(x) \leftrightarrow \bar{\chi}(0, x)P_{+} \quad \zeta(x) \leftrightarrow P_{-}\chi(0, x)$$

Note: The chirally rotated framework is only chosen for technical convenience. Using the above dictionary any standard SF correlator can be easily translated to this rotated framework (for an even number of fermions)
Orbifold techniques have been used to implement the standard SF conditions for Ginsparg-Wilson quarks (Taniguchi '04). Here:

- start with standard lattice action for a single massless quark flavour

\[ S_f[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\psi}(x) D_N \psi(x), \quad D_N = 1 - A(A^\dagger A)^{-1/2}, \quad A = 1 - a D_W \]

where

\[ \psi(x_0 + 2T, x) = -\psi(x), \quad \bar{\psi}(x_0 + 2T, x) = -\bar{\psi}(x) \]

- introduce a reflection \((R^2 = id)\)

\[ R : \psi(x) \rightarrow i \gamma_0 \gamma_5 \psi(-x_0, x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(-x_0, x)i \gamma_0 \gamma_5 \]

- the gauge field is extended to \([-T, T]\) and then periodically continued:

\[ U_k(-x_0, x) = U_k(x_0, x), \quad U_0(-x_0 - a, x)^\dagger = U_0(x) \]
• Decompose fields into even and odd with respect to $R$,

\[ R\psi_\pm = \pm\psi_\pm, \quad R\bar{\psi}_\pm = \pm\bar{\psi}_\pm \]

• even/odd fields satisfy the boundary conditions at $x_0 = 0$

\[ (1 \mp i\gamma_0\gamma_5)\psi_\pm(0, x) = 0 \quad \bar{\psi}_\pm(0, x)(1 \mp i\gamma_0\gamma_5) = 0 \]

• and with complementary projectors at $x_0 = T$, due to antiperiodicity:

\[ (1 \pm i\gamma_0\gamma_5)\psi_\pm(T, x) = 0 \quad \bar{\psi}_\pm(T, x)(1 \pm i\gamma_0\gamma_5) = 0 \]

• $[D_N, R] = 0$

\[ \Rightarrow S_f[\psi, \bar{\psi}, U] = S_f[\psi_+ + \psi_-, \bar{\psi}_+ + \bar{\psi}_-, U] = S_f[\psi_+, \bar{\psi}_+, U] + S_f[\psi_-, \bar{\psi}_-, U] \]

$\Rightarrow$ the functional integral factorises!
interpret even and odd fields as quark flavours

\[ \chi = \sqrt{2} \left( \frac{\psi_-}{\psi_+} \right), \quad \bar{\chi} = \sqrt{2} \left( \bar{\psi}_- \bar{\psi}_+ \right) \]

functional integral:

\[ \int \prod_{-T \leq x_0 < T} d\psi(x) d\bar{\psi}(x) e^{-S_f[\psi, \bar{\psi}, U]} \propto \int \prod_{0 \leq x_0 \leq T} d\chi(x) d\bar{\chi}(x) e^{-\frac{1}{2} S_f[\chi, \bar{\chi}, U]} \]

equivalent to theory in the interval \([0, T]\) with boundary conditions

\[ Q_+ \chi(x)|_{x_0=0} = 0, \quad Q_- \chi(x)|_{x_0=T} = 0, \]
\[ \bar{\chi}(x) Q_+|_{x_0=0} = 0, \quad \bar{\chi}(x) Q_-|_{x_0=T} = 0 \]

\[ S_f[\chi, \bar{\chi}, U] = a^4 \sum_{-T < x_0 \leq T} \bar{\chi}(x) D_N \chi(x) = 2a^4 \sum_{0 \leq x_0 \leq T} \bar{\chi}(x) D_N \chi(x), \]
Explicit reduction to the interval using alternative set-up:

- By a slight $O(a)$ deformation of the orbifold, a block structure (in time) of the Wilson-Dirac operator is obtained and inherited by the overlap operator

$\Rightarrow$ the Neuberger operator in the interval is simply obtained by using the corresponding orbifolded Wilson-Dirac kernel:

$$D_N = 1 - A(A^\dagger A)^{-1/2}, \quad A = 1 - aD_W$$

with the kernel $D_W$,

$$aD_W \chi(x) = -U(x, 0)P_- \chi(x + a\hat{0}) + (K\psi)(x) - U(x - a\hat{0})^\dagger P_+ \chi(x - a\hat{0}),$$

where we have set $\chi(x) = 0$ for $x_0 < 0$ and $x_0 > T$, and

$$K = 1 + \frac{1}{2} \sum_{k=1}^{3} \left\{ a(\nabla_k + \nabla_k^*) \gamma_k - a^2 \nabla_k^* \nabla_k \right\} + \delta_{x_0,0} i\gamma_5 \tau^3 P_- + \delta_{x_0,T} i\gamma_5 \tau^3 P_+$$
Symmetries and Counterterms

In a massless theory in finite volume the identification of flavour and chiral symmetries is a mere convention!

• take the standard Schrödinger functional with projectors $P_\pm$ as SU(2) flavour symmetric reference basis

• in the rotated SF, the SU(2) flavour symmetry is realised à la Ginsparg-Wilson:

\[
\begin{align*}
\gamma_5 \tau^{1,2} D_N + D_N \gamma_5 \tau^{1,2} &= D_N \gamma_5 \tau^{1,2} D_N \\
\tau^3 D_N - D_N \tau^3 &= 0
\end{align*}
\]

Note that the flavour algebra closes $[\Gamma_5 = \gamma_5(1 - aD_N)]$:

\[
\hat{T}^1 = \Gamma_5 \tau^2 / 2, \quad \hat{T}^2 = -\Gamma_5 \tau^1 / 2, \quad \hat{T}^3 = \tau^3 / 2, \quad [\hat{T}^a, \hat{T}^b] = i \epsilon^{abc} \hat{T}^c
\]
• Chiral symmetry is broken by the SF boundary conditions: expect the standard GW relation is violated by terms which decrease exponentially with the distance from the boundaries.

• Form of non-singlet chiral symmetries:

\[ [\tau^{1,2}, D_N] \neq 0, \quad \{\gamma_5 \tau^3, D_N\} \neq aD_N\gamma_5\tau^3D_N \]

expect: both flavour components of \( D_N \) become equal and the GW relation holds up to corrections which should decrease exponentially with the distance from the boundaries.

• GW versions of parity and time reversal, e.g.:

\[ P : \chi(x) \rightarrow i\gamma_0\gamma_5\tau^3\chi(\tilde{x}), \quad \tilde{x} = (x_0, -x), \quad D_NP + PD_N = D_NPD_N \]

• No extra dimension 3 counterterm (unlike Wilson); for most correlation functions expect contribution by 2 \( O(a) \) boundary counterterms (analogous to \( c_t, \tilde{c}_t \) in standard SF).
Free propagator and Neuberger operator

The orbifold construction allows for a straightforward computation of the free propagator in time-momentum representation:

\[
\tilde{S}_N(p; x_0, y_0) = \frac{1}{2(T+a)} \sum_{k=-T/a}^{T/a+1} \tilde{D}_N^{-1}(p) \left\{ e^{ip_0(x_0-y_0)} - i\gamma_0\gamma_5\tau^3 e^{ip_0(x_0+y_0+a)} \right\},
\]

\[
p_0 = p_0(k) = \frac{\pi(k+\frac{1}{2})}{T+a}, \quad a\tilde{D}_N(p) = 1 - \frac{1 - \frac{1}{2}a^2\hat{p}^2 - ia\hat{p}}{1 + \frac{1}{2}a^4 \sum_{\mu<\nu} \hat{p}^2_{\mu}\hat{p}^2_{\nu}}^{1/2}.
\]

I checked numerically that flavour and parity identities hold exactly, e.g.

\[
\gamma_5 \tilde{D}_N(p; x_0, y_0)^{(1)} + \tilde{D}_N(p; x_0, y_0)^{(2)} \gamma_5 = \sum_{z_0=0}^{T/a} \tilde{D}_N(p; x_0, z_0)^{(1)} \gamma_5 \tilde{D}_N(p; z_0, y_0)^{(2)}
\]

The GW relation is broken by terms which vanish exponentially with the distance from the boundaries (as expected)
Conclusions and Outlook

• Successful implementation of chirally rotated SF boundary conditions for even number of GW quarks

• In the continuum limit the chirally rotated SF with an even number of massless GW quarks is equivalent to the standard SF;
  – parity and flavour symmetries are exact on the lattice!
  – solution is technically simple: just requires the insertion of the corresponding Wilson kernel into the Neuberger relation
  – The determinant is real and non-negative

• Construction applies directly to Domain Wall Quarks, technically simple, no obstruction for mass term of Pauli-Villars fields.