Automatic generation of vertices for the Schrödinger functional

Shinji Takeda, Ulli Wolff

Humboldt Universität zu Berlin

Lattice 2007
Regensburg, July 30 - August 4, 2007
Automatic generation of vertices on the lattice

- Lüscher and Weisz algorithm ’86
  - Applicable for closed loops (gauge action) only

- Bottom up algorithm Hart and et. al. ’05
  - Generalization of LW algorithm
  - Applicable for any parallel transporters (not only for closed loop)
    - Fermion, HQET with smeared action are OK
  - For infinite or all periodic lattice (but not SF)
Bottom up algorithm

Hart and et. al. ’05

- **Coordinate space**
  \[ U(x, \mu) = \exp\left[g_0 q_\mu(x)\right] \]

- **Arbitrary parallel transporter along** $\mathcal{L}$
  \[ \alpha = (\mu, x) \]
  \[ P[\mathcal{L}; q] = \sum_{r=0}^{\infty} \frac{g^r_0}{r!} P_r[\mathcal{L}; q], \quad P_r = \sum_{a_1, \ldots, a_r} C_{a_1 \cdots a_r} \sum_{i=1}^{N_r} q_1^{a_1} \cdots q_r^{a_r} f_i \]
  
  - **Color factor:** $C_{a_1 \cdots a_r} = T_{a_1} \cdots T_{a_r}$
  
  - **Amplitude:** $f_i \in R$

- **List:** $L_i^{(r)} = (\alpha_1^{a_1}, \ldots, \alpha_r^{a_r}; f_i)$
  \[ \{L_i^{(r)} \mid i = 1, \ldots, N_r\} \iff V_{\alpha_1 \cdots \alpha_r}^{(r)} \text{: reduced Vertex (no color index)} \]

- **Elementary building block:** Link variable

  \[ U(x, \mu) = \sum_{r=0}^{\infty} \frac{g^r_0}{r!} \sum_{a_1 \cdots a_r} T_{a_1} T_{a_2} \cdots T_{a_r} q_1^{a_1} \cdots q_r^{a_r} 1 \]

  - **List:** $L^{(r)} = (\alpha, \cdots, \alpha; 1)^{r \text{ elements}}$
Bottom up algorithm

Hart and et. al. ’05

- Parallel transporter \( P'P'' = P \)

- Multiplication of set: \( \{L_i\}_P \times \{L_j\}_P \rightarrow \{L_k\}_P \)

\[
P_r = \sum_{s=0}^{r} \frac{r!}{s!(r-s)!} P'_s P''_{r-s}
\]

\[
= \sum_{a_1\ldots a_r} C_{a_1\ldots a_r} \sum_{s=0}^{r} \frac{r!}{s!(r-s)!} \sum_{i=1}^{N'_s} \sum_{j=1}^{N''_{r-s}} q_{a_1} \ldots q_{a_s} q_{a_{s+1}} \ldots q_{a_r} f_1 f''_j
\]

- Relabeling: \( \{\alpha^1_i, \ldots, \alpha^s_i, \alpha^1_j, \ldots, \alpha^{r-s}_j\} \rightarrow \{\alpha^1_k, \ldots, \alpha^s_k, \alpha^{s+1}_k, \ldots, \alpha^r_k\} \)

- Amplitude part: \( \frac{r!}{s!(r-s)!} f'_i f''_j \rightarrow f_k \)

- List: \( L'_i \times L''_j \rightarrow L_k = (\alpha^1_k, \ldots, \alpha^r_k; f_k) \)

- Implemented in **Python** script language to deal with the list operations.
Extension to Schrödinger functional

- Schrödinger functional (SF)  
  Lüscher et al. '92, '94

Presence of background field \( V(x, \mu) \)

\[
V(x, 0) = 1, \quad V(x, k) = V(x_0)
\]

\[
\Downarrow
\]

\[
U(x, \mu) = V(x, \mu)e^{g_0q_\mu(x)}
\]

Main difficulty: **Color factor** becomes complicated by \( V \)

Usual lattice  

\[
C_{a_1a_2a_3 \ldots} = T_{a_1}T_{a_2}T_{a_3} \cdots \rightarrow T_{a_1}V(x, \mu)T_{a_2}V^{-1}T_{a_3} \cdots
\]

\[
P_r = \sum_{a_1, \ldots, a_r} C_{a_1 \ldots a_r} \sum_{i=1}^{N_r} q_{a_1}^{\alpha_i} \cdots q_{a_r}^{\alpha_i} f_i 
\]  

SF

\[
\sum_{a_1, \ldots, a_r} \sum_{i=1}^{N_r} C_{a_1 \ldots a_r}^{i} q_{a_1}^{\alpha_i} \cdots q_{a_r}^{\alpha_i} f_i
\]
Color factor for SF case

- Property of back ground field $V$:
  
  \[ V = e^{i \sum_i h_i H_i} : \text{diagonal} \implies V I_a V^{-1} = I_a e^{i \phi_a(x_0)} \]
  
  ($H_i \in \text{Cartan subalgebra}, \quad [H_i, I_a] = \mu_{ia} I_a, \quad \phi_a = \sum_i \mu_{ia} h_i$)

- $\phi_a(x_0) = \phi_a(0) + x_0 \psi_a \cdot \text{linear on } x_0$

- Expression for any color factor (for order $r$)

  \[ C^k = I_{a_1} \cdots I_{a_{s-1}} V I_{a_s} V^{-1} I_{a_{s+1}} \cdots I_{a_r} \]
  
  \[
  = \left[ I_{a_1} \cdots I_{a_r} V^{A_k} e^{i \mathcal{E} B_k} \right] \left( e^{i \sum_{u=1}^r \psi_{au} C_k^{(u)} + \phi_{au} D_k^{(u)}} \right)
  \]
  
  \( 3 \times 3 \text{ matrix} \quad U(1) \text{ phase factor} \)

- List \((A_k, B_k, C_k, D_k)\) are integer value and characterize the color factor $C^k$

- Information of lattice size $L$, $T$ and back ground field are encoded in $V$, $\mathcal{E}$, $\psi$ and $\phi$
Algorithm for color factor

- Multiplication of $C_k \leftarrow C_i C_j$

\[
(A, B, C, D) \leftarrow (A', B', C', D') \times (A'', B'', C'', D''),
\]

$A \leftarrow A' + A''$

$B \leftarrow B' + B'' + \Delta t A''$

$C \leftarrow (C'_1, \ldots, C'_s), C''_1 + 2B' + \Delta t D''_1, \ldots, C''_{r-s} + 2B' + \Delta t D''_{r-s})$

- $s$ elements

- $r-s$ elements

$D \leftarrow (D'_1, \ldots, D'_s), D''_1 + 2A', \ldots, D''_{r-s} + 2A')$

- $s$ elements

- $r-s$ elements

- $r$ elements

$\Delta t$: time difference between $P'$ and $P''$

- The algorithm is suited for symbolic operation and easily implemented in Python

List: $L^{(r)}_k = (\alpha^1_k, \ldots, \alpha^r_k, A_k, B_k, C_k, D_k; f_k)$

- Applicable for any parallel transporter

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Check of algorithm

- Physical quantity for check: one-loop SF coupling
  \( gg \) vertex \((r = 2)\): inverse propagator

- Gauge actions
  - Plaquette gauge action (by hand Lüscher et al. ’94)
  - Improved gauge action (by hand Takeda et al. ’03)

- \( ggg \) vertex \((r = 3)\) of plaquette gauge action
  (by hand Weisz ’96)

  \boxed{\text{Consistent with hand derived vertices}}

- \( gggg \) vertex \((r = 4)\)
  two-loop SF coupling (by hand Narayanan and Wolff ’95)

  \(\implies\) Future
Application I: \( L = T + sa \) lattice \((s = \pm 1)\)

- Motivated by considering staggered fermion on SF
  \( T = \text{odd} \), due to Dirichlet BC for time direction

Here discuss only gauge part, but not the staggered fermion

\[ \implies \text{talk by P.P. Rubio for staggered fermion part} \]

- Continuum limit with fixed \((T + sa)/L = 1\)

- (Modified) Background field

\[
\begin{align*}
B(0) &= C, \quad B(T) = C', \\
B(x_0) &= f \left( x_0 - \frac{T}{2} \right) + \frac{C + C'}{2}, \text{ for } a < x_0 < T - a.
\end{align*}
\]

where \( f \) is determined from EOM numerically

- Boundary counter term: \( c_t^{(0)} = \frac{2}{2+s} \)

- We extend our algorithm to apply to arbitrary time dependent phase \( \phi(x_0) \).
Application I: $L = T + sa$ lattice ($s = \pm 1$)

- Relative deviation of step scaling function

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1^{(k)}(a/L)u + O(u^2),$$

$$\sigma(u) = \bar{g}^2(2L), \quad u = \bar{g}^2(L),$$

$k = 0 (k = 1)$: tree (one-loop) level $O(a)$ improved case

![Graphs showing relative deviation vs. a/L for different s values](image)
Application II: $\Lambda$ for improved gauge actions

- **Improved gauge actions** $(c_0 + 8c_1 + 16c_2 + 8c_3 = 1)$

- **Ratio of $\Lambda$ parameter:**
  \[
  \Lambda_{\text{improved}} / \Lambda_{\text{plaquette}} = \frac{\Lambda_{\text{improved}}/\Lambda_{\text{SF}}}{\Lambda_{\text{plaquette}}/\Lambda_{\text{SF}}}
  \]

<table>
<thead>
<tr>
<th>improved action</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>Our results</th>
<th>previous results</th>
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<tr>
<td>Iwasaki</td>
<td>$-0.331$</td>
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<td>5.294(4)</td>
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<td>0</td>
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<td>1308</td>
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<td>Wilson RG</td>
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<td>$-0.170$</td>
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<td>67.97(9)</td>
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</table>

previous results: Iwasaki et al., ’84, Weisz et al., ’84, Sakai et al., ’00
Concluding remarks

- We provide an algorithm to produce the vertices for the SF and implement it in the Python script language.

**Key point** Lattice background field $V(x, \mu)$ is diagonal

- Check of the algorithm:
  - One-loop SF coupling ($gg$ vertex) for the plaquette and improved gauge action
  - $ggg$ vertex for the plaquette gauge action are correctly reproduced.

- Applications:
  - $L = T + a$ lattice (motivated by staggered fermion on SF),
    Phase $\phi_a(x_0)$ has arbitrary time dependence
  - $\Lambda$ parameter for improved gauge actions (six links)
Future

- To obtain further confidence, 2-loop SF coupling has to be done

- Applications
  - Fermion part
  - 2-loop calculation for SF correlation functions (mass, Z factors, and so on)
  - HQET with smearing