Unquenching effects on the coefficients of the Lüscher-Weisz action

Zh. Hao\(^a\), G.M. von Hippel\(^{b,*}\), R.R. Horgan\(^c\), Q.J. Mason\(^c\), H.D. Trottier\(^{a,d}\)

*HPQCD collaboration*

\(^a\) Simon Fraser University, Department of Physics, Burnaby, BC, V5A 1S6, Canada

\(^b\) Department of Physics, University of Regina, Regina, SK, S4S 0A2, Canada

\(^c\) DAMTP, CMS, University of Cambridge, Cambridge CB3 0WA, U.K.

\(^d\) TRIUMF, Vancouver, BC, V6T 2A2, Canada
Unquenching and Improvement

Crucial ingredients for accurate simulations:

- **Perturbative improvement**: include loop effects via improvement coefficients in the action
- **Unquenching**: include fermion loops in simulations
Unquenching and Improvement

Crucial ingredients for accurate simulations:

- **Perturbative improvement**: include loop effects via improvement coefficients in the action
- **Unquenching**: include fermion loops in simulations

**Discrepancy**: unquenched simulations are being done with actions improved only for glue loops

- increased scaling violations seen in some unquenched simulations
Unquenching and Improvement

- Crucial ingredients for accurate simulations:
  - **Perturbative improvement**: include loop effects via improvement coefficients in the action
  - **Unquenching**: include fermion loops in simulations

- **Discrepancy**: unquenched simulations are being done with actions improved only for glue loops
  - increased scaling violations seen in some unquenched simulations

- **Needed**: $\mathcal{O}(\alpha_s N_f)$ contributions to the improvement coefficients
Unquenching and Improvement

- Crucial ingredients for accurate simulations:
  - **Perturbative improvement**: include loop effects via improvement coefficients in the action
  - **Unquenching**: include fermion loops in simulations

- **Discrepancy**: unquenched simulations are being done with actions improved only for glue loops
  - increased scaling violations seen in some unquenched simulations

- **Needed**: $\mathcal{O}(\alpha_s N_f)$ contributions to the improvement coefficients

- **Here**: computation of $\mathcal{O}(\alpha_s N_f a^2)$ Lüscher-Weisz on-shell improvement for dynamical improved staggered quarks
On-Shell Improvement

- Original Symanzik improvement program: patch action to remove $O(a^2)$ artifacts from Green's functions

- **Problem:** in gauge theories, Green's functions are not gauge-invariant
On-Shell Improvement

- Original Symanzik improvement program: patch action to remove $\mathcal{O}(a^2)$ artifacts from Green's functions
- **Problem:** in gauge theories, Green's functions are not gauge-invariant
- **Solution:** Lüscher-Weisz on-shell improvement, remove $\mathcal{O}(a^2)$ artifacts from spectral quantities only.
Original Symanzik improvement program: patch action to remove $\mathcal{O}(a^2)$ artifacts from Green's functions

**Problem**: in gauge theories, Green's functions are not gauge-invariant

**Solution**: Lüscher-Weisz on-shell improvement, remove $\mathcal{O}(a^2)$ artifacts from spectral quantities only.

Write action as

$$S = \sum_x \left\{ (1 - 8(c_1 + c_2)) \sum_{\mu \neq \nu} \langle 1 - P_{\mu\nu} \rangle + 2c_1 \sum_{\mu \neq \nu} \langle 1 - R_{\mu\nu} \rangle + \frac{4}{3} c_2 \sum_{\mu \neq \nu \neq \rho} \langle 1 - T_{\mu\nu\rho} \rangle \right\}$$

Expand spectral quantities as

$$q_i = \bar{q}_i + a_i (\mu a)^2 + d_{ij} c_j (\mu a)^2 + \mathcal{O} ((\mu a)^4)$$

to get improvement conditions

$$d_{ij} c_j = -a_i$$
On-Shell Improvement

- Original Symanzik improvement program: patch action to remove $O(a^2)$ artifacts from Green's functions.

- **Problem:** in gauge theories, Green's functions are not gauge-invariant.

- **Solution:** Lüscher-Weisz on-shell improvement, remove $O(a^2)$ artifacts from spectral quantities only.

- Write action as

  \[
  S = \sum_x \left\{ (1 - 8(c_1 + c_2)) \sum_{\mu \neq \nu} \langle 1 - P_{\mu\nu} \rangle + 2c_1 \sum_{\mu \neq \nu} \langle 1 - R_{\mu\nu} \rangle + \frac{4}{3} c_2 \sum_{\mu \neq \nu \neq \rho} \langle 1 - T_{\mu\nu\rho} \rangle \right\}
  \]

- Expand spectral quantities as

  \[
  q_i = \tilde{q}_i + a_i (\mu a)^2 + d_{ij} c_j (\mu a)^2 + O((\mu a)^4)
  \]

  to get improvement conditions

  \[
  d_{ij} c_j = -a_i
  \]

- Linear equation $\Rightarrow$ can decompose into gluonic and fermionic part.
Twisted Periodic Boundary Conditions

- With periodic b.c., hard to find spectral quantities that are accessible perturbatively
- Also, there are perturbative IR divergences for pure glue
Twisted Periodic Boundary Conditions

- With periodic b.c., hard to find spectral quantities that are accessible perturbatively.
- Also, there are perturbative IR divergences for pure glue.
- Use twisted b.c. to regulate IR divergences and provide a natural mass scale.
  - Defined by $(\mu, \nu = 1, 2)$

\[
U_\mu(x + L\hat{\nu}) = \Omega_\nu U_\mu(x)\Omega_\nu^{-1}
\]
\[
\Psi(x + L\hat{\nu}) = \Omega_\nu \Psi(x)\Omega_\nu^{-1}
\]

where quarks pick up an additional “smell” index.
Twisted Periodic Boundary Conditions

- With periodic b.c., hard to find spectral quantities that are accessible perturbatively
- Also, there are perturbative IR divergences for pure glue
- Use twisted b.c. to regulate IR divergences and provide a natural mass scale
  - Defined by \( (\mu, \nu = 1, 2) \)

\[
U_\mu(x + L\hat{\nu}) = \Omega_\nu U_\mu(x) \Omega_\nu^{-1}
\]

\[
\Psi(x + L\hat{\nu}) = \Omega_\nu \Psi(x) \Omega_\nu^{-1}
\]

where quarks pick up an additional “smell” index.

- Momenta are quantised in units of \( m = \frac{2\pi}{3L} \) in the twisted directions: \( p_\nu = mn_\nu, n \neq (0, 0) \mod 3 \) for gluons.
- Kaluza-Klein theory in \((z, t)\) plane has stable massive “mesons” \( A (n = (0, 1), (1, 0)) \) and \( B (n = (1, 1)) \), providing spectral quantities.
Mass of the A Meson

- Simplest spectral quantity: renormalised mass of the $A$ meson

- At tree level: $m_A^{(0)} = m + \mathcal{O}((ma)^4)$

- At one loop, two fermionic diagrams (bubble and tadpole) contribute

- Improvement contributes

$$\Delta_{\text{imp}} \frac{m_A^{(1)}}{m} = (c_1^{(1)} - c_2^{(1)}) (ma)^2 + \mathcal{O}((ma)^4)$$
Three-Point Coupling (I)

- Next simplest independent spectral quantity is cross-section $A + A \rightarrow A + A$ at $B$ threshold
- Can be reduced to $AAB$ coupling $\lambda$ defined by

$$\lambda = g_0 \sqrt{Z(k)Z(p)Z(q)} \Gamma^{1,2,j}(k, p, q)$$

for appropriate $k, p, q$ and $e$
- At tree level, $\lambda^{(0)} = -8m + O((ma)^4)$
- At one-loop level, 8 fermionic diagrams contribute
- Need to be careful with $z, t$ integration contours to pick up correct analytic continuation
- Improvement contributes

$$\Delta_{\text{imp}} \frac{\lambda^{(1)}}{m} = 4(9c_1^{(1)} - 7c_2^{(1)})(ma)^2 + O((ma)^4)$$
Three-Point Coupling (II)
Small-Mass Expansions

- Need to isolate $O(a^2)$ lattice artifacts
- Two mass scales in fermionic case: $m$ and $m_q$, $m_q/m > C$ needed
Need to isolate $\mathcal{O}(a^2)$ lattice artifacts

Two mass scales in fermionic case: $m$ and $m_q, m_q/m > C$ needed

First expand in powers of $ma$ at fixed $m_q a$:

$$Q(ma, m_q a) = a_0^{(Q)}(m_q a) + a_2^{(Q)}(m_q a)(ma)^2 + \mathcal{O}((ma)^4)$$
**Small-Mass Expansions**

- Need to isolate $\mathcal{O}(a^2)$ lattice artifacts
- Two mass scales in fermionic case: $m$ and $m_q$, $m_q/m > C$ needed
- First expand in powers of $ma$ at fixed $m_qa$:

\[
Q(ma, m_qa) = a_0^{(Q)}(m_qa) + a_2^{(Q)}(m_qa)(ma)^2 + \mathcal{O}((ma)^4)
\]

- Use physical constraints (continuum limit, $\beta$-function, IR behaviour) to determine form of $a_i^{(Q)}(m_qa)$:

\[
a_0^{(Q)}(m_qa) = b_0^{(Q)} \log(m_qa) + a_{0,0}^{(Q)} + a_{0,2}^{(Q)}(m_qa)^2 + \mathcal{O}((m_qa)^4)
\]

\[
a_2^{(Q)}(m_qa) = \frac{a_{2,2}^{(Q)}}{(m_qa)^2} + a_{2,0}^{(Q)} + \left( a_{2,2}^{(Q)} + b_{2,2}^{(Q)} \log(m_qa) \right)(m_qa)^2 + \mathcal{O}((m_qa)^4)
\]

- Chiral limit $m_q \to 0 \Rightarrow a_{2,0}^{(Q)}$ is relevant term.
Automated Perturbation Theory

- We use the asqtad action for the quarks
- A $\bar{\psi} A^3 \psi$ vertex occurs in the diagrams
Automated Perturbation Theory

- We use the asqtad action for the quarks
- A $\bar{\psi} A^3 \psi$ vertex occurs in the diagrams
- Automated generation of Feynman rules is a must
  - Feynman rules are derived using automated manipulations (in Python, Maple or C++)
  - For details see hep-lat/0411026 or hep-lat/0310044
Automated Perturbation Theory

- We use the asqtad action for the quarks
- A $\bar{\psi} A^3 \psi$ vertex occurs in the diagrams
- Automated generation of Feynman rules is a must
  - Feynman rules are derived using automated manipulations (in Python, Maple or C++)
  - For details see hep-lat/0411026 or hep-lat/0310044
- We need the second derivatives of the self-energy
Automated Perturbation Theory

- We use the asqtad action for the quarks
- A $\bar{\psi} A^3 \psi$ vertex occurs in the diagrams
- Automated generation of Feynman rules is a must
  - Feynman rules are derived using automated manipulations (in Python, Maple or C++)
  - For details see hep-lat/0411026 or hep-lat/0310044
- We need the second derivatives of the self-energy
- So is use of automatic differentiation methods
  - Operator overloading is used to implement Taylor algebra
  - For details see physics.comp-ph/0506222
  - Another approach uses finite differences
Results: A meson mass

![Graph 1: Self-Energy $m_A^{(1)} / m$](image1)

- $am_q = 0.2$

![Graph 2: Self-Energy $g_2(am_q)$](image2)

- $am_q$ range from 0 to 1.0
Results: Three-point coupling (I)

$Triple-Gluon: g_o(a m_q)$

$-\log(a m_q)/(3\pi^2) + c$

$Triple-Gluon: g_o(a m_q) + \log(a m_q)/(3\pi^2)$
Results: Three-point coupling (II)

\begin{align*}
\text{Triple-Gluon: } & g_2(am_q) \\
\text{Triple-Gluon: } & g_2(am_q) + \frac{1}{120\pi^2(am_q)^2} + c
\end{align*}
Results: Summary

Solving improvement conditions gives:

\[
\begin{align*}
    c_1^{(1)} &= -0.025218(4) + 0.00486(13) N_f \\
    c_2^{(1)} &= -0.004418(4) + 0.00126(13) N_f
\end{align*}
\]

Shift from unquenched value is surprisingly large for \( N_f = 3 \).

May explain increased scaling violations observed

Should be included in future simulations

Possible to include in (re-)analysis of existing simulations