The phase diagram of QCD at finite isospin density

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QCD at finite isospin density

- Special case of the physically relevant situation,
- Platform to assess limitations of various numerical approaches to finite baryon density,
- Provides a rich range of physical phenomena:
  - pion gas at low temperature and density,
  - quark gas at high temperature,
  - Bose condensation of charged pions at large density.
Transition from hadronic to quark degrees of freedom
⇒ due to a large density of a conserved charge.

- system does not carry baryon number,
- chemical potentials of two degenerate quarks $u$ and $d$ are equal in magnitude, $|\mu_I|/2$, but opposite in sign,
- accessible by lattice simulations.

Positivity of the theory is guaranteed by

$$\tau_1 \gamma_5 D \gamma_5 \tau_1 = D^\dagger.$$

Using this positivity and QCD inequalities [Son & Stephanov]:
⇒ symmetry breaking must be driven by $\langle \bar{\psi} i \gamma_5 \tau_1, 2\psi \rangle$,
i.e. $\pi^- \sim \bar{u} \gamma_5 d$, $\pi^+ \sim \bar{d} \gamma_5 u$ states.
At small isospin densities one can use chiral perturbation theory

\[ \mathcal{L} = \frac{1}{4} f_{\pi}^2 \text{Tr}[\partial_\mu \Sigma \partial_\mu \Sigma^\dagger - 2m_{\pi}^2 \text{Re} \Sigma] \]

where \( \Sigma \in \text{SU}(2) \) is the matrix pion field:

- \( \mu_I \) breaks \( \text{SU}(2)_{L+R} \rightarrow \text{U}(1)_{L+R} \),
- no additional low energy constant needed (to leading order),
- interesting physics for \( m_{\pi} < \mu_I < m_{\rho} \).

Effective potential can be minimised as a function of \( \mu_I \) using

\[ \overline{\Sigma} = \cos \alpha + i(\tau_1 \cos \phi + \tau_2 \sin \phi) \sin \alpha, \]

- flavour rotation angle \( \phi \) irrelevant.
Two distinct regimes can be identified.

- \(|\mu_I| < m_\pi\):
  - no pion can be excited,
  - \(\bar{\Sigma} = 1\), i.e. \(\langle \bar{u}u + \bar{d}d \rangle = 2\langle \bar{\psi}\psi \rangle_0\)
  - normal QCD vacuum.

- \(|\mu_I| \geq m_\pi\):
  - \(\pi^-\) particles can be excited,
  - a Bose condensate of \(\pi^-\) forms where \(\langle \bar{u}\gamma_5d \rangle \neq 0\),
  - chiral condensate rotates into pion condensate as a function of \(\mu_I\),
  - \(\pi^-\) becomes massless, \(\pi^+, \pi^0\) remain massive.

At \(|\mu_I| > m_\rho\) chiral perturbation theory breaks down.
Energies \( m \) to excite a pion from the vacuum at low temperature:

\[
\begin{align*}
\pi^- & \quad \text{at } \mu_I = m_\pi \quad \text{the } \pi^- \text{ Bose condense.}
\end{align*}
\]
'Equation of state' (EoS) : density as a function of isospin chemical potential:

\[ \rho_I = \frac{Q}{V} = \rho_I(\hat{\mu}_I) \]

where \( \hat{\mu}_I = \frac{\mu_I}{T} \).

Accessible from \textit{canonical simulations} is the free energy \( F(Q) = -\ln Z_C(Q) \) and its derivative

\[ F(Q) - F(Q - 1) \xrightarrow{V \to \infty} \frac{dF}{d\rho_I} = \mu_I. \]
EoS for free bosons, i.e. pions at low density:

\[
\rho(\hat{\mu}, \hat{m}) = \frac{T^3}{2\pi^2} \int_0^{+\infty} d\hat{p} \hat{p}^2 \left( \frac{1}{e^{(\omega - \hat{\mu})} - 1} - \frac{1}{e^{(\omega + \hat{\mu})} - 1} \right)
\]

where \( \omega = \sqrt{\hat{p}^2 + \hat{m}^2} \).
EoS for interacting Bose gas, i.e. Bose condensate at $T = 0$:

$$\rho_I = f^2_\pi \mu_I \left(1 - \left(\frac{m_\pi}{\mu_I}\right)^4\right)$$
EoS for interacting Bose gas at low $T$:

⇒ interaction pushes critical density down
EoS for a massless, free Fermi gas via the pressure:

\[
\frac{P(\mu_I) - P(\mu_I = 0)}{T^4} = \frac{1}{2} \left( \frac{\mu_I}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_I}{T} \right)^4.
\]
Lattice simulation details

- \( N_f = 4 + 4 \), i.e. 2 staggered fermions on \( 8^3 \times 4 \) at \( am = 0.14 \):
  \[ \Rightarrow \text{deconfinement transition at } \mu = 0 \text{ is } 1^{\text{st}} \text{ order} \]

- Temperature ranges between \( \frac{1}{2} T_c \leq T \leq T_c \),

- Pion mass \( am_\pi \) changes only by few percent:
  \[ \Rightarrow m_\pi / T \sim \text{constant} \]

- Combine 68 ensembles at 6 values of \( \mu \) up to \( \mu / T \leq 4 \) with Ferrenberg-Swendsen reweighting.
Free energy

\[ \frac{\Delta F}{T} \sim \frac{\mu}{T} \]

- Bose condensate
- Free Fermi gas

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Free energy

\[ \frac{\Delta F}{T} - \frac{\mu}{T} \]

\[ m_\pi / T \]

\[ Q \]

\[ \Delta F/T \sim \mu/T \]

Bose condensate

Free Fermi gas

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Free energy

\[ \Delta F/T \sim \mu/T \]

- Bose condensate
- Free Fermi gas

QCD at finite isospin density
Free energy

\[ \Delta F/T = \mu/T \]

- Bose condensate
- Free Fermi gas
Free energy

Numerical results

Free energy

Phase diagram

Bose condensation

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Free Fermi gas

Bose condensate

QCD at finite isospin density
Free energy

\[ \Delta F/T - \mu/T \]

\[ m_\pi/T \]

\[ \Delta F/T \sim \mu/T \]

Bose condensate
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Free Fermi gas

\( \frac{m_\pi}{T} \)

\( \Delta F/T - \mu/T \)

\( Q \)

Bose condensate

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Free energy

Bose condensate
Free Fermi gas

\[ \Delta F/T - \mu T \]

\[ m_\pi/T \]

\[ Q \]

\[ 0 \quad 50 \quad 100 \quad 150 \quad 200 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

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Free energy

\[ \Delta F/T \sim \mu/T \]

Bose condensate
Free Fermi gas

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Free energy

The graph shows the free energy $\Delta F/T$ as a function of $Q$, with $m_\pi/T$ on the y-axis and $Q$ on the x-axis. The graph includes two curves:

- **Dashed blue line** represents the Bose condensate.
- **Solid red line** represents the Free Fermi gas.

The free energy is given by $\mu/T$, where $\mu$ is the chemical potential.
Free energy

\[ \Delta F/T \sim \mu/T \]

\[ m_\pi/T \]

- Bose condensate
- Free Fermi gas

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Phase diagram

Plasma phase

Hadronic phase

BEC

\( \mu / T \)

\( \beta \)

\( m_\pi / T \)

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QCD at finite isospin density
Phase diagram

- Plasma phase
- Hadronic phase
- Bose condensation (BEC)

QCD at finite isospin density

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Phase diagram

Plasma phase

Hadronic phase

\[ \beta \]

\[ \mu/T \]

\[ m_\pi/T \]

Gluodynamics (MC)

Quadratic fit

Quartic fit

BEC

QCD at finite isospin density
Phase diagram

![Graph showing phase transitions in QCD at finite isospin density. The graph plots the plasma phase transition and Bose condensation against the chemical potential ($\mu/T$) and inverse temperature ($\beta$). Quadratic and quartic fits are shown, with data points indicating the critical points.]
Phase diagram

Plasma phase

- GC MC
- Quadratic fit
- Quartic fit
- Critical point

Free energy
Phase diagram
Bose condensation

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Bose condensation

Transition from BEC to the Fermi gas:
⇒ measure pion susceptibility $\chi_{\pi^-}$

![Graph showing the transition from Bose-Einstein condensation (BEC) to the Fermi gas, with measures of pion susceptibility $\chi_{\pi^-}$ for different isospin densities $\mu$.]
Bose condensation

Rescale to recover universal behaviour:

![Graph showing Bose condensation](image-url)
Universality class of the 3d \textit{xy}-model:

\[ \chi_{\pi} = \mu \]

\[ \mu = 0.48 \quad \mu = 0.50 \quad \mu = 0.55 \quad \mu = 0.60 \]

\[ 3d \text{ xy, } L=12 \Rightarrow \text{good agreement} \]
Reweighting from $\mu = 0$ ensembles alone gives unreliable results.
Reweighting from $\mu = 0$ ensembles alone gives unreliable results.

Average sign of the determinant:
We determined the **EoS and the phase diagram** of $N_f = 4 + 4$ QCD **at finite isospin density** and finite temperature.

We exposed the two mechanisms at work:
- Bose condensation at high density,
- deconfinement at high temperature.

**Implications for the baryonic density case.**