Velocity-dependent potentials at $O(1/m^2)$

Yoshiaki Koma

*Numazu College of Technology*

— Lattice 2007, Regensburg, 30 July 2007 —

In collaboration with: Miho Koma and Hartmut Wittig (Mainz)
Mt. Fuji, a view from Numazu College of Technology
Potential NRQCD

- Effective field theory of QCD for heavy quarks
  [Brambilla, Pineda, Soto & Vairo (’99–)]

- Potential picture for heavy quarkonium
  ⇒ static potential + relativistic corrections with $1/m$ expansion

- Effective Hamiltonian up to $O(1/m^2)$ [Pineda & Vairo (’01)]

\[
H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V^{(0)}(r) + \frac{1}{m_1}V^{(1,0)}(r) + \frac{1}{m_2}V^{(0,1)}(r)
\]
\[
+ \frac{1}{m_1^2} V^{(2,0)}(r) + \frac{1}{m_2^2} V^{(0,2)}(r) + \frac{1}{m_1 m_2} V^{(1,1)}(r) + O(1/m^3)
\]
⇒ potentials need to be determined nonperturbatively

- static potential $V^{(0)}$: well known
- relativistic corrections at $O(1/m)$: determined (M. Koma’s talk)
- relativistic corrections at $O(1/m^2)$: to be determined

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Relativistic corrections

Structure of the relativistic corrections at $O(1/m^2)$

\[
\begin{align*}
V^{(2,0)} &= V^{(2,0)}_{SD} + V^{(2,0)}_{SI} \quad (V^{(2,0)} = V^{(0,2)}), \\
V^{(1,1)} &= V^{(1,1)}_{SD} + V^{(1,1)}_{SI}
\end{align*}
\]

⇒ spin-dependent potentials are equivalent to the expression of [Eichten&Feinberg(’79,’81)] (except some matching factors)
[de Forcrand&Stack(’85), Michael&Rakow(’85), Michael(’86), Huntley&Michael(’87), Campostrini,Moriarty&Rebbi(’86,’87), Bali,Schilling&Wachter(’97)] . . . [Koma&Koma(’07)]

⇒ spin-independent potentials contain velocity-dependent potential of [Barchielli,Brambilla,Montaldi&Prosperi(’88,’90)]
[Bali,Schilling&Wachter(’97)] . . . [Koma,Koma&Wittig, this talk]

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Velocity-dependent potentials

▶ Spin-independent potentials [Pineda & Vairo (01)]

\[ V_{SI}(r) = \frac{1}{m_1^2} \left( \frac{1}{2} \{ p_1^2, V_{p^2}^{(2,0)}(r) \} + \frac{V_{l^2}^{(2,0)}(r)}{r^2} l_1^2 + V_r^{(2,0)}(r) \right) + (1 \rightarrow 2) \]

\[ + \frac{1}{m_1 m_2} \left( -\frac{1}{2} \{ p_1 \cdot p_2, V_{p^2}^{(1,1)}(r) \} - \frac{V_{l^2}^{(1,1)}(r)}{2r^2} (l_1 \cdot l_2 + l_2 \cdot l_1) + V_r^{(1,1)}(r) \right) \]

▶ Velocity-dependent potentials

\[ V_{p^2}^{(2,0)}(r), \ V_{l^2}^{(2,0)}(r), \ V_{p^2}^{(1,1)}(r), \ V_{l^2}^{(1,1)}(r) \]

⇒ relation to Barchielli et al.'s parametrization

\[ V_{p^2}^{(2,0)} = V_d - \frac{2}{3} V_e, \quad V_{l^2}^{(2,0)} = V_e, \quad V_{p^2}^{(1,1)} = -V_b + \frac{2}{3} V_c, \quad V_{l^2}^{(1,1)} = -V_c \]

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Velocity-dependent potentials

- Nonperturbative expression

\[
V_b(r) = -\frac{1}{3} \int_0^\infty dt \, t^2 \langle\langle E(\vec{0}, 0) \cdot E(\vec{r}, t)\rangle\rangle_c
\]

\[
V_d(r) = \frac{1}{6} \int_0^\infty dt \, t^2 \langle\langle E(\vec{0}, 0) \cdot E(\vec{0}, t)\rangle\rangle_c
\]

\[
\left( \frac{r_i r_j}{r^2} - \frac{\delta_{ij}}{3} \right) V_c(r) = \int_0^\infty dt \, t^2 \{ \langle\langle E^i(\vec{0}, 0) E^j(\vec{r}, t)\rangle\rangle_c - \frac{\delta_{ij}}{3} \langle\langle E(\vec{0}, 0) \cdot E(\vec{r}, t)\rangle\rangle_c \}
\]

\[
\left( \frac{r_i r_j}{r^2} - \frac{\delta_{ij}}{3} \right) V_e(r) = -\frac{1}{2} \int_0^\infty dt \, t^2 \{ \langle\langle E^i(\vec{0}, 0) E^j(\vec{0}, t)\rangle\rangle_c - \frac{\delta_{ij}}{3} \langle\langle E(\vec{0}, 0) \cdot E(\vec{0}, t)\rangle\rangle_c \}
\]

⇒ (connected) electric field strength correlator on the quark-antiquark source

\[
\langle\langle E_i E_j \rangle\rangle_c = \langle\langle E_i E_j \rangle\rangle - \langle\langle E_i \rangle\rangle \langle\langle E_j \rangle\rangle, \quad \langle\langle E_i E_j \rangle\rangle = \langle E_i E_j \rangle_W / \langle W \rangle
\]

⇒ attach the electric field operators to the quark line,
   same side or opposite side, with various orientation

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Numerical procedure

- Evaluate field strength correlators on the Polyakov loop correlation function (using the multilevel algorithm)
- Fit to spectral representation of field strength correlator

\[ \langle\langle E_i(0, 0) E_j(0, \tau)\rangle\rangle_c = \]
\[ 2 \sum_{n=1}^{\infty} \langle 0 | E_i(0) | n \rangle \langle n | E_j(0) | 0 \rangle e^{-(\Delta E_{n0}) \frac{T}{2}} \cosh \left( (\Delta E_{n0}) \left( \frac{T}{2} - \tau \right) \right) + O(e^{-(\Delta E_{10}) T}) \]

where \( \Delta E_{n0}(r) = E_n(r) - E_0(r) \), \( E_0(r) = V^{(0)}(r) \)

- finite-\( T \) effect is automatically taken into account in the fit
- error term \( O(e^{-(\Delta E_{10}) T}) \) is negligible for a reasonably large \( T \)
- no numerical integration, no extrapolation

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Lattice setup

- Wilson gauge action

- Gauge coupling, lattice volume, statistics

<table>
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<th>$\beta = 6/g^2$</th>
<th>$a$ [fm]</th>
<th>$V = L^3 T$</th>
<th>$N_{tsl}$</th>
<th>$N_{iupd}$</th>
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<td>5</td>
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</tr>
</tbody>
</table>

NEC SX8@RCNP Osaka University

- Electric field strength operator

$$g a^2 F_{4i} \equiv (U_{4i} - U_{4i}^\dagger)/(2i) \quad \text{(traceless, two-leaf modification)}$$

- Huntley-Michael factor [Huntley&Michael(’87)]

$$Z_{F_{\mu\nu}} = \langle PP^* \rangle / \langle \text{Re} U_{\mu\nu} \rangle_{PP^*} \quad \text{(cancel most of self energies at } O(g^2))$$

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Electric field strength correlator

Same side correlator (if Wilson loop is used)

\[
\langle \langle E_1(0, t_1)E_1(0, t_2) \rangle \rangle = \frac{\langle E_1(0, t_1)E_1(0, t_2) \rangle_W}{\langle W \rangle}
\]

\[
\langle E_1(0, t_1)E_1(0, t_2) \rangle_W = \langle \frac{-1}{2i} [U_{41}^\dagger(0, t_1) - U_{41}(0, t_1)] \frac{-1}{2i} [U_{41}^\dagger(0, t_2) - U_{41}(0, t_2)] \rangle_W
\]
Electric field strength correlator

- Opposite side correlator (if Wilson loop is used)

\[ \langle\langle E_1(0, t_1)E_1(r, t_2)\rangle\rangle = \frac{\langle E_1(0, t_1)E_1(r, t_2)\rangle_W}{\langle W\rangle} \]

\[ \langle E_1(0, t_1)E_1(r, t_2)\rangle_W = \langle \frac{-1}{2i} [U_{41}^\dagger(0, t_1) - U_{41}(r, t_1)] \frac{1}{2i} [U_{41}(0, t_2) - U_{41}^\dagger(r, t_2)] \rangle_W \]
Electric field strength correlator: result

▶ e.g.) $r/a = 5$ at $\beta = 5.85$ on the $18^324$ lattice

- statistical errors are quite small

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Electric field strength correlator: result

▶ e.g.) \( r/a = 5 \) at \( \beta = 5.85 \) on the \( 18^3 24 \) lattice

- statistical errors are quite small
- fitting to the spectral rep. of the FSC works nicely

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Velocity-dependent potentials: result

- $V_b(r)$ and $V_d(r)$

- clean data up to 0.9 fm
- good scaling behavior

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Velocity-dependent potentials: result

$V_c(r)$ and $V_e(r)$

- clean data up to 0.9 fm
- good scaling behavior
BBMP relation: result

**BBMP relation** [Barchielli,Brambilla,Montaldi&Prosperi(’88,’90)]

Poincaré invariance of field strength correlators (in the continuum limit)

\[ V_b + 2V_d = -\frac{1}{2}V^{(0)} + \frac{r}{6}V^{(0)'}, \quad V_c + 2V_e = -\frac{r}{2}V^{(0)'} \]

![Graphs showing the BBMP relation with data points for different values of \( \beta \).]
Summary

- The velocity-dependent relativistic corrections to the static potential at $O(1/m^2)$

- Clean data up to 0.9 fm
  - multilevel, $PP^*$, transfer matrix theory work well

- Good scaling behavior
  - Better renormalization of field strength operators could do better

- BBMP relation, satisfied

- Comparison with models and phenomenology, to be done
Appendix

Huntley-Michael factor \( Z_{F_{\mu \nu}} = \frac{\langle PP^* \rangle}{\langle \text{Re} U_{\mu \nu} \rangle_{PP^*}} \)

- dependence on \( r \) and relative orientation to the \( q-\bar{q} \) axis, \( \vec{r} = (r, 0, 0) \)

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