Non-perturbative renormalization of four-quark operators and $B_K$ with SF scheme in quenched domain-wall QCD

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for CP-PACS Collaboration

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Plan of talk

- Definition of $B_K$ and its renormalization factors.

- The strategy of our study

- SF correlation function
  - Definition of non-perturbative renormalization factors
    - Alpha collaboration ’05

- Simulation results
  - The results of RGI $Z_{B_K}$ & RGI $\hat{B}_K$
    - The scaling behavior of SSF at $g^2(L) = 3.480$

- Summary
Definition of $B_K$ and its renormalization factor

The kaon $B$ parameter: $B_K$

$$B_K \equiv \frac{\langle \bar{K}^0 | \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma^\mu (1 - \gamma_5) d | K^0 \rangle}{\frac{8}{3} \langle \bar{K}^0 | \bar{s} \gamma^\mu (1 - \gamma_5) d | 0 \rangle \langle 0 | \bar{s} \gamma^\mu (1 - \gamma_5) d | K^0 \rangle}$$

CP-PACS has calculated $B_K$ in quenched domain-wall fermion \textsuperscript{2001}\
⇒ We want to derive NP renormalized $B_K$

Renormalization factors for $B_K$

$$Z_{B_K}(a\mu) = \frac{Z_{VV+AA}(g_0, a\mu)}{Z_A^2(g_0)} = \frac{Z_{VA+AV}(g_0, a\mu)}{Z_V^2(g_0)}$$

DWF has a good chiral symmetry on the lattice

$Z_V(g_0) = Z_A(g_0)$ has been shown by CP-PACS \textsuperscript{2003}

$Z_{VV+AA}(g_0, a\mu) = Z_{VA+AV}(g_0, a\mu)$

⇒ this relation will be checked later
Our strategy

\[ Z_{B_K}(g_0) = Z_{V_A+A_V}^{PT}(\infty, \mu_{\text{max}}) \underbrace{Z_{V_A+A_V}^{NP}(\mu_{\text{max}}, \mu_{\text{min}})}_{\text{III}} \underbrace{Z_{B_K}^{NP}(g_0, a\mu_{\text{min}})}_{\text{I}} \]

- Lattice bare \( B_K^{(0)} \) with DWF
  \( \Rightarrow \) Renormalization group invariant (RGI) \( \hat{B}_K \)

Three steps for RGI \( \hat{B}_K \) using SF scheme.

1. \( B_K^{(0)} \Rightarrow B_K^{\text{(SF)}}(a\mu_{\text{min}}) \) at hadronic scale
   suppress lattice artifact \( O(a\mu) : a\mu \ll 1 \)

2. Non-pert. RG running : \( B_K^{\text{(SF)}}(\mu_{\text{min}}) \Rightarrow B_K^{\text{(SF)}}(\mu_{\text{max}}) \)
   Already calculated by Alpha collab. for \( O_{V_A+A_V} '05 \)
   \( \Rightarrow O_{V_A+A_V} \) has no mixing problem even for wilson fermion.

3. Pert.trans. to RGI : \( B_K^{\text{(SF)}}(\mu_{\text{max}}) \Rightarrow \hat{B}_K \)

II. III. are regularization independent part.

Our target: I. \( Z_{V_A+A_V}(g_0, a\mu_{\text{min}}) \) with DWF.
SF set up and correlation function \textit{Alpha ’05}

\begin{align*}
\mathcal{O}_{ij} &= a^6 \sum_{\bar{x}\bar{y}} \bar{\zeta}_i(\bar{x}) \Gamma \zeta_j(\bar{y}) \\
\mathcal{O}_{VA+AV} &= (\bar{\psi}_1 \gamma_\mu \psi_2)(\bar{\psi}_3 \gamma_\mu \gamma_5 \psi_4) + (\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2)(\bar{\psi}_3 \gamma_\mu \psi_4) \\
\mathcal{O}_{ij} &= a^6 \sum_{\bar{x}\bar{y}} \bar{\zeta}_i(\bar{x}) \Gamma \zeta_j(\bar{y})
\end{align*}

**Correlation function** (\textit{Alpha collaboration ’05})

\[
\mathcal{F}^\pm_{[\Gamma_A, \Gamma_B, \Gamma_C]}(x_0) = \frac{1}{L^3} \langle \mathcal{O}_{21}[\Gamma_A] \mathcal{O}_{45}[\Gamma_B] \mathcal{O}_{VA+AV}^\pm(x) \mathcal{O}_{53}'[\Gamma_C] \rangle
\]

**Orbifolding construction for SF with DWF** Y.Taniguchi ’04,’06
In order to remove the divergence due to the boundary fields

\[ f_1 = -\frac{1}{2L^6}\langle O'_{12}[\gamma_5]O_{21}[\gamma_5]\rangle, \quad k_1 = -\frac{1}{2L^6}\langle O'_{12}[\gamma_k]O_{21}[\gamma_k]\rangle \]

Three specific cases

**Scheme 1**  \[ h_{1}^{\pm}(x_0) = \frac{\mathcal{F}_{[\gamma_5,\gamma_5,\gamma_5]}^{\pm}(x_0)}{f_1^{3/2}} \]

**Scheme 3**  \[ h_{3}^{\pm}(x_0) = \frac{\mathcal{F}_{[\gamma_5,\gamma_k,\gamma_k]}^{\pm}(x_0)}{f_1^{3/2}} \]

**Scheme 7**  \[ h_{7}^{\pm}(x_0) = \frac{\mathcal{F}_{[\gamma_5,\gamma_k,\gamma_k]}^{\pm}(x_0)}{f_1^{1/2}k_1} \]

Renormalization condition

\[ Z_{VA+AV; s}(g_0, a\mu)h_{s}^{\pm}(x_0; g_0) = h_{s}(x_0, g_0)^{\pm}|_{g_0=0}, \quad s = 1, 3, 7 \]

Renormalization factors

\[ Z_{VA+AV; s}^{\pm} = \frac{h_{s}^{\pm}(x_0; g_0)|_{g_0=0}}{h_{s}^{\pm}(x_0; g_0)} \]

⇒ The ratio of continuum tree level and lattice correlation function
Simulation parameters

- Same parameters as in previous CP-PACS calculation of $B_K$ in Phys.Rev. D64(2001)114506

- Domain-wall fermion with $M = 1.8$ and $N_s = 16$

- Iwasaki gauge action at $\beta = 2.6, \beta = 2.9, \beta = 3.2$ (new, not in the paper) corresponding to $a^{-1} \sim 2, a^{-1} \sim 3, a^{-1} \sim 4$ GeV

- Renormalization factor $Z_{VA+AV}(a\mu)$

  - $1/\mu_{\text{min}} = 2L_{\text{max}}$

  - Def. of $L_{\text{max}}$: $\bar{g}_{SF}^2(L_{\text{max}}) = 3.480$

  - The $Z_{VA+AV}$ at $\beta = 2.6, \beta = 2.9, \beta = 3.2$

- Need to fine tune $\beta$ such that $aN = 2L_{\text{max}} = 1.498r_0$

<table>
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<th>$L_{\text{max}}/a$</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
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<td>1000</td>
<td>1018</td>
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<td>556</td>
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Simulation results

The results of $Z_V$ and $Z_{VA+AV; s}$

\[Z_V\]

\[Z_{VA+AV; s}(x_0 = \frac{16}{2}, a\mu_{\text{min}})\]

\[Z_{BK}(a\mu_{\text{min}}) = \frac{Z_{VA+AV; s}(x_0 = \frac{16}{2}, a\mu_{\text{min}})}{Z_V(x_0 = \frac{16}{2}, a\mu_{\text{min}})} = 1.2708(53)\]

We take the renormalization factor at $x_0 = N_T/2$
Remind you the our strategy

- Lattice bare $B_K^{(0)}$ with DWF

$\Rightarrow$ Renormalization group invariant (RGI) $\hat{B}_K$

\[
Z_{B_k}(g_0) = Z_{VA+AV}^{PT}(\infty, \mu_{\text{max}}) Z_{VA+AV}^{NP}(\mu_{\text{max}}, \mu_{\text{min}}) Z_{B_k}^{NP}(g_0, a\mu_{\text{min}})
\]
Results for RGI $Z_{B_K}$ and $\hat{B}_K$ (quench)

Fitting form

$$\hat{Z}_{B_K}(\beta) = a_1 + b_1(\beta - 3) + c_1(\beta - 3)^2$$

interpolated at $\beta = 2.6, \beta = 2.9, \beta = 3.2$

$\hat{B}_K = 0.773$ (7)(preliminary) : Scheme 1

$\hat{B}_K = 0.760$ (7)(preliminary) : Scheme 3

$\hat{B}_K = 0.779$ (8)(preliminary) : Scheme 7

$\hat{B}_K = 0.786(31)$ : RBC

$\hat{B}_K = 0.789(46)$ : Alpha

Error is smaller

Scaling behaviour is better

Error of each data is smaller

The smallest lattice spacing is finer
Conversion of RGI $B_K$ to that in $\overline{\text{MS}}$ scheme

- RG running from RGI to $\overline{\text{MS}}$ NDR $\mu = 2\text{GeV}$ in NLO

$$B_K^{\overline{\text{MS}}} (NDR, \mu) = \left[ \frac{g^2_{\overline{\text{MS}}} (\mu)}{4\pi} \right]^{\frac{\gamma_0^+}{2b_0}} \exp \left[ \int_0^{g_{\overline{\text{MS}}} (\mu)} dg \left( \frac{\gamma^+}{\beta(g)} - \frac{\gamma_0^+}{b_0 g} \right) \right] \hat{B}_K$$

- Gauge coupling

$$\Lambda_{\overline{\text{MS}}} = \mu (b_0 g^2_{\overline{\text{MS}}} )^{-\frac{b_1}{2b_0^2}} \exp \left[ - \frac{1}{2b_0 g^2_{\overline{\text{MS}}}} \right] \exp \left[ - \int_0^{g_{\overline{\text{MS}}}} dg \left( \frac{1}{\beta_{\overline{\text{MS}}} (0)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right) \right]$$

- $\Lambda_{\overline{\text{MS}}} = 0.586(48)/r_0$ Necco & Sommer ’02
Comparison our results of $B_K(\overline{\text{MS}}, 2\text{GeV})$ with previous results

$B_K(\text{NDR,} 2\text{GeV}) = 0.557(5)(^{+4}_{-10})$ (preliminary)

$B_K^{\overline{\text{MS}}}(\text{NDR,} 2\text{GeV}) = 0.567(4)$ : CP-PACS(perturbative ren.)

$B_K^{\overline{\text{MS}}}(\text{NDR,} 2\text{GeV}) = 0.563(21)$ : RBC

$B_K^{\overline{\text{MS}}}(\text{NDR,} 2\text{GeV}) = 0.573(34)$ : Alpha

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Further checks

- **Continuum limit of Step Scaling Function**

\[
\Sigma_{VA+AV}\,s(u, L_{max}/a) = \frac{Z_{VA+AV}\,s(g_0, a/2L_{max})}{Z_{VA+AV}\,s(g_0, a/L_{max})}
\]

- **Chiral symmetry breaking effects in renormalization factors**

\[
Z_{VA+AV} = Z_{VV+AA}
\]
Continuum limit of SSF at \( g^2(L) = u = 3.480 \)

\[
\sigma_{VA+AV;s}(u) = \lim_{a \to 0} \Sigma_{VA+AV;s}(u, a/L_{\text{max}}) = \left. \frac{Z_{VA+AV;s}(g_0, a/2L_{\text{max}})}{Z_{VA+AV;s}(g_0, a/L_{\text{max}})} \right|_{g^2=u}
\]

Boundary effects are large at \( M = 1.8 \)
Results at $N_S = 8$ and $N_S = 32$

No discrepancies in $N_S$
**SOLUTION 1**  :  results at $M = 1.4$

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SOLUTION 2: Results at $M = 1.8$ with a tree-level improve

\[ Z_{VA+AV;\sigma}(g_0, a\mu) h_s^\pm(x_0; g_0) = h_s(x_0, g_0)^\pm |_{g_0=0} \]

CONT $\Rightarrow h_s(x_0, g_0)^\pm |_{g_0=0}$

Scaling violation is improved
Continuum limit is consistent with Alpha’s
The continuum limit of SSF for $Z_{B_K}$

$$
\Sigma_{B_K}(u, L_{\text{max}}/a) = \frac{Z_{B_K};s(g_0, a/2L_{\text{max}})}{Z_{B_K};s(g_0, a/L_{\text{max}})}
$$

Results at $M = 1.4$ and $M = 1.8$ are consistent

$O(a)$ errors are partly cancelled in $Z_{B_K}$
Check the chiral symmetry breaking

**WT Identity**

\[ \delta q_1 = -i\gamma_5 \tilde{q}_1, \quad \delta \zeta_1 = i\gamma_5 \tilde{\zeta}_1, \quad \delta \zeta'_1 = i\gamma_5 \tilde{\zeta}'_1 \]

\[ \langle O_{VA+AV} O[\zeta]\rangle_S = \langle O_{VV+AA} O'[\zeta]\rangle_S \]

\[ \Rightarrow Z_{VV+AA} = Z_{VA+AV} \]

**For the domain-wall case**

\[ S \rightarrow S + Y : Y \text{ is the chiral symmetry breaking term} \]

\[ \langle O_{VA+AV} O[\zeta]\rangle_S = \langle O_{VV+AA} O'[\zeta]\rangle_{S+Y} \neq \langle O_{VV+AA} O'[\zeta]\rangle_S \]

\[ \langle O_{VV+AA} O[\zeta]\rangle_S \Leftrightarrow \langle O_{VA+AV} O'[\zeta]\rangle_S \]

We investigate if \( Z_{VV+AA} = Z_{VA+AV} \) holds or not.
The results of $Z_{VA+AV;1}$ and $Z_{VV+AA;1}$ ($N_s = 16$)

Good agreement: $Z_{VA+AV;1}$ and $Z_{VV+AA;1}$

$Z_{B_K} = \frac{Z_{VA+AV}}{Z_V^2}$: no problem
Summary

- Determined the NP renormalization factors of $B_K$
- Bare $B_K^{(0)}$ on lattice with DWF $\Rightarrow$ RGI $\hat{B}_K$
- Investigating the chiral breaking effects
  - Find $Z_{VA+AV} = Z_{VV+AA}$ in DWQCD
- RGI $\hat{B}_K = 0.773(7)$ (preliminary)
- $\overline{\text{MS}}$ $B_K(\text{NDR, } 2\text{GeV}) = 0.557(5)(^{+4}_{-10})$ (preliminary)
- Consistent with the previous results
- Check the scaling behavior of SSF
Extra
Time direction are extended doubled; \( N_T \to 2N_T \) Taniguchi ’04

Gauge configuration: copied into negative time region

\[
U_k(\vec{x}, x_0) = U_k(\vec{x}, -x_0), \quad U_0(\vec{x}, x_0) = U_0(\vec{x}, -x_0),
\]

Fermion field: orbifolding projection \( x_0 \leftrightarrow -x_0 \)

\[
\psi(\vec{x}, -x_0, s) = \gamma_0 \gamma_5 P Q \psi(\vec{x}, x_0, s)
\]

⇒ projecting out the following symmetric subspace

\[
\bar{\Pi}_- \psi(x, s) = 0, \quad (\bar{\psi} \bar{\Pi}_-)(x, s) = 0, \quad \bar{\Pi}_\pm = \frac{1 \pm \gamma_0 \gamma_5 P Q}{2}
\]

Boundary fields obey a projection condition

\[
\bar{P}_- \psi(\vec{x}, 0, s) = 0, \quad (\bar{\psi} \bar{P}_-)(x, s) = 0, \quad \bar{P}_\pm = \frac{1 \pm \gamma_0 \gamma_5 P Q}{2}
\]

For physical quark

\[
P_+ q(x)|_{x_0=0} = 0, \quad P_- q(x)|_{x_0=N_T}
\]

⇒ the SF boundary condition
Investigation of the chiral symmetry breaking

- Correlation function under chiral rotation for first flavour

\[ F_{A,B,C}^{\pm}(x_0) = \frac{1}{L^3} \langle O_{21}[\Gamma_A]O_{45}[\Gamma_B]O_{VA+AV}^\pm(x)O'_{53}[\Gamma_C]\rangle_S \]

\[ = \frac{1}{L^3} \langle iO_{21}[\Gamma_A\gamma_5]O_{45}[\Gamma_B]O_{VV+AA}^\pm(x)O'_{53}[\Gamma_C]\rangle_{S+Y} = \tilde{F}_{A,B,C}^{\pm}(x_0) \]

- \( q_1 = -i\gamma_5\tilde{q}_1, \quad \zeta_1 = i\gamma_5\tilde{\zeta}_1, \quad \zeta'_1 = i\gamma_5\tilde{\zeta}'_1 \)

- \( Y \) is the chiral symmetry breaking term.

- Z factor for the parity odd:

\[ Z_{VA+AV;1}(g_0, a\mu) = \frac{h_1(x_0;g_0=0)}{h_1(x_0;g_0)}, \quad h_1(x_0) = \frac{F_{[\gamma_5\gamma_5\gamma_5]}[\gamma_5\gamma_5\gamma_5]}{f_1^{3/2}} \]

- Z factor for the parity even:

\[ Z_{VA+AV;1}(g_0, a\mu) = \frac{\tilde{h}_1(x_0;g_0=0)}{\tilde{h}_1(x_0;g_0)}, \quad \tilde{h}_1(x_0) = \frac{\tilde{F}_{[\gamma_5\gamma_5\gamma_5]}[\gamma_5\gamma_5\gamma_5]}{f_1^{3/2}} \]

If there are no contribution of \( Y \) (chiral symmetry breaking), we can find

\[ Z_{VA+AV} = Z_{VA+AV} \]