SFB Training
The Lanczos approach as a general principle
12th January 2012 — Wuppertal

Task 1 \textit{Lanczos}

1. Implement the Lanczos procedure

\begin{verbatim}
function \[T,V\] = lanczos(A,v,m)
\end{verbatim}

which computes the Lanczos basis \( V_{m+1} = [v_1 | v_2 | \cdots | v_{m+1}] \) of the Krylov subspace \( \mathcal{K}_m(A,v) \) and the associated tridiagonal Hessenberg matrix \( T_{m+1,m} \).

2. Visualize \( V_m^\dagger V_m - I \) using the MATLAB function \texttt{imagesc} for different values of \( m \) and the test matrices that you find in the \texttt{data} folder.

3. Compute the eigenvalues of \( T_m \) in each iteration of the Lanczos procedure and plot their history as in the following example.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{lanczos_eigenvalue_history.png}
\caption{Lanczos eigenvalue history}
\end{figure}
4. Kaniel-Paige theory predicts the numerical loss of orthogonality of $V_m$ as soon as an eigenvalue of $T_m$ converged to an eigenvalue of $A$. Verify this statement computationally using a diagonal matrix $A$ with entries distributed between 1 and 100 and a random start vector $v$.

**Task 2**  *Conjugate Gradients*

Use the MATLAB built-in implementation of the Conjugate Gradient (CG) method `pcg` to solve the Wilson-Dirac system of linear equations

$$D_\kappa x = b$$

using the normal equations $D^T_\kappa D_\kappa x = D^T_\kappa b$ for a point-source right hand side $b$ and varying values of $\kappa$ approaching $\kappa_c$ for $D_\kappa = I - \kappa \cdot D$. Plot the convergence history of all runs in a log-scale plot. You can find operators and associated values of $\kappa_c$ in the folder `data`.

*Optional:* Implement the action of $D^T_\kappa D_\kappa$ in a separate function and call `pcg` using a function handle and compare the respective runtimes.

**Task 3**  *Matrix function evaluation*

Use the Lanczos process to approximate

$$\text{sign}(Q) \cdot b, \quad Q = \gamma_5 \cdot D_\kappa$$

for $\kappa = \frac{1}{2} \kappa_c$. You can find operators and associated values of $\kappa_c$ in the `data` folder. Do you have an idea how you could measure the accuracy of your approximation?