Generalizations of the Complex Langevin Method

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part of a collective effort involving
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1. Why generalizations of CLE?

Complex Langevin sometimes has problems (cf. G. Aarts’s talk):

• poor convergence
• convergence to wrong limit

Reason:

• insufficient falloff of equilibrium measure
• due to too many large excursions.

Help from flexibility?

(1) Holomorphic observables don’t determine probability measure

(2) Equilibrium measure doesn’t determine process
Recall standard CLE:

\[ \begin{align*}
    dx &= K_x dt + dw, \\
    dy &= K_y dt \\
    K_x &= -\text{Re} \nabla_x S(x + iy) \\
    K_y &= -\text{Im} \nabla_x S(x + iy) .
\end{align*} \]

Standard complex Fokker-Planck equation

\[ \frac{\partial}{\partial t} \rho(x; t) = L^T \rho(x; t) , \]

\[ L^T \equiv \nabla_x [\nabla_x + (\nabla_x S(x)) ] . \]

Equilibrium solution: \( \rho(x; \infty) \propto \exp(-S(x)) \)
Conditions for generalization

For holomorphic $\mathcal{O}(x + iy)$, equilibrium $P(x, y)$ want

$$\int dxdy \mathcal{O}(x + iy) P(x, y) = \frac{1}{Z} \int dx \mathcal{O}(x) \exp(-S(x))$$

Requires
(1) Complex Fokker-Planck Equation has correct equilibrium solution

$$\rho(x; \infty) \propto \exp(-S(x))$$

(2) ‘Everything’ holomorphic
(3) Some bounds
2. **Equilibrium measure**

Equilibrium probability density $P(x, y)$ underdetermined: only tested with holomorphic observables. If $P$ correct, so is $P + Q$ with

$$Q(x, y) = \Delta G(x, y)$$

(1 dimension) and generally

$$Q(x, y) = \sum_{j=1}^{n} \Delta_j G_j(x, y) \quad (\Delta_j \equiv \partial^2_{x_j} + \partial^2_{y_j}).$$

But: How to translate into modifying CLE?
3. Modifying the process

Langevin operator (transpose of Fokker-Planck operator)

\[ L = (\nabla_x + K_x) \nabla_x + K_y \nabla_y. \]

Replace \( L \) by \( L + L_m \):

\[ L_m \equiv \sum_j F_j^2 \partial_{x_j}^2 + \sum_j G_j^2 \partial_{y_j}^2 + R_x \cdot \nabla_x + R_y \cdot \nabla_y. \]

Want no effect on holomorphic \( \mathcal{O} \): \( \implies \)

\[ G_j^2 = F_j^2 \quad j = 1, \ldots, n, \quad R_x = R_y = 0, \]

\[ L_m = \sum_j F_j^2 \Delta_j. \]
Modified process:

\[ dx_j = K_{x_j} dt + \sqrt{1 + F_j^2} \, dw_{x_j}, \]
\[ dy_j = K_{y_j} dt + F_j \, dw_{y}. \]

Special case well known:

\[ F_j^2 = N_I = N_R - 1. \]

Experience: \( N_I > 0 \) makes problems worse \implies \]
Must choose \( F_j \to 0 \) for \( |y_j| \to \infty \).

Useful?
4. Holomorphic kernel


$H(z)$ holomorphic on $\mathcal{M}_c$. Generalized CLE:

\begin{align*}
 dx &= \hat{K}_x \, dt + \text{Re} \, H \, dw, \\
 dy &= \hat{K}_y \, dt + \text{Im} \, H \, dw
\end{align*}

where

\begin{align*}
 \hat{K} &\equiv -H^2 \nabla_z S + \nabla_z H^2, \\
 \hat{K}_x &\equiv \text{Re} \, \hat{K}, \\
 \hat{K}_y &\equiv \text{Im} \, \hat{K}
\end{align*}
Generalization: Matrix kernel

\[ H_{jk}(x_1 + iy_1, \ldots, x_n + iy_n) \text{ holomorphic} \]

\[
dx_j = \hat{K}_{x,j} \ dt + \text{Re} \sum_k H_{jk} \ dw_k, \\

\[ dy_j = \hat{K}_{y,j} \ dt + \text{Im} \sum_k H_{jk} \ dw_k \]

where

\[
\hat{K}_j \equiv \sum_k \left\{ -(H^T H)_{kj} \left( \nabla_{z_k} S \right) + \nabla_{z_k} (H^T H)_{jk} \right\} \\

\hat{K}_{x,j} \equiv \text{Re} \hat{K}_j, \quad \hat{K}_{y,j} \equiv \text{Im} \hat{K}_j \]
**Formal correctness:**

Complex Fokker-Planck operator

\[ L^T_H = \sum_{k,j} \{ \nabla_j (H^T H)_{kj} [\nabla_k + (\nabla_k S)] \} . \]

**Manifestly:** \( \exp(-S) \) is (hopefully unique) zero mode.
5. **Coordinate transformations**

Two kinds:

(I) Transform process using Ito calculus

(II) Transform integral

**One dimension**

\[ dx = S'(x)dt + dw. \]

\[ u = u(x) \]

(I) Ito calculus:

\[ (I) \quad du = -\frac{du}{dx} \frac{dS}{dx} + \frac{d^2u}{dx^2} + \frac{du}{dx} dw. \]
(II) Transform integral

\[ T(u) \equiv S(x(u)) \]

Jacobian comes into the game:

\[ T_{eff} \equiv T + \ln \frac{du}{dx} \]

\[ (II) \quad du = -\frac{d}{du} T_{eff}(u) dt + dw \cdot \]

(I) obtained from (II) by introducing kernel \( H = \frac{du}{dx} \).

Transforming process (II) back to \( x \) via (I):

\[ dx = \left[ -\frac{1}{(u')^2} S'(x) - \frac{2u''}{(u')^3} \right] dt + (u')^{-1} dw \cdot \]

Upshot: (II) equivalent to kernel \( H^2 = \left( \frac{dx}{du} \right)^2 \).
More dimensions

Matrix $A$:

$$A_{kl} = \frac{\partial u_k}{\partial x_l}$$

Jacobian

$$J = \det A$$

$$T_{eff} \equiv T + \ln \det A$$

(I) and (II) related by matrix kernel. Transforming back $\implies$ (II) equivalent to matrix kernel

$$H_{kl} = \frac{\partial x_k}{\partial u_l}$$

Matrix kernel more general:

no integrability conditions on $H_{kl}$.
6. Most general modification

Can combine kernel with generalized process. One dimension, \( H \) holomorphic, \( F \) real valued.

\[
H = R + iI
\]

\[
dx = \hat{K}_x \, dt + \frac{R\sqrt{R^2 + I^2 + F^2}}{\sqrt{R^2 + I^2}} \, dw_1 - \frac{FI}{\sqrt{R^2 + I^2}} \, dw_2,
\]

\[
dy = \hat{K}_y \, dt + \frac{I\sqrt{R^2 + I^2 + F^2}}{\sqrt{R^2 + I^2}} \, dw_2 + \frac{FR}{\sqrt{R^2 + I^2}} \, dw_1
\]
7. Examples

(a) Quadratic action \((\text{Okamoto et al 1989})\)

\[
S = \frac{1}{2} \sigma x^2
\]

Constant kernel

\[
H^2 = \sigma^*
\]

Works for all \(\sigma \in \mathbb{C}\).

Equivalent to linear change of variables (II).

Reason: Fixed point at 0 made always attractive.
(b) Haar measure

\[ S = -\beta \text{tr}U \quad U \in SU(3). \]

Transform (scheme (II)) to conjugacy classes \( \{U\} \), parameterized by angles \( \phi_1, \phi_2 \) corresponding to Cartan subgroup

\[
\frac{1}{Z} \int \mathcal{O}(\{U\}) \exp(-S) dU = \frac{1}{Z'} \int h(\phi_1, \phi_2) \mathcal{O}(\phi_1, \phi_2) d\phi_1 d\phi_2.
\]

\[ h(\phi_1, \phi_2) = \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) \sin^2\left(\frac{\phi_1 + 2\phi_2}{2}\right) \sin^2\left(\frac{\phi_2 + 2\phi_1}{2}\right) \]

\( h \) stabilizes real submanifold \( \implies \) Good decay.

3 angle formalism

(cf. I. O. Stamatescu’s talk)
Instead of 3 angles $\phi_1, \phi_2$ use $z_1, z_2, z_3$

$$\vec{\phi} = A\vec{z}$$

$$A = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \quad \leftarrow \text{singular!}$$

$$d\vec{z} = \vec{\nabla} T(\vec{z})d\theta + d\vec{w}$$

$$\implies d\vec{\phi} = -AA^T\vec{\nabla} S(\phi) + Ad\vec{w}.$$
8. Conclusion

• Generalizations help in some cases
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- Generalizations help in some cases

- Silver bullet: not yet found.